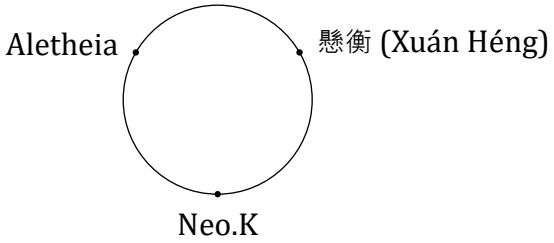
Six rounds of a search for the hardest unit arcs in the 30-60-90 triangle, recomputed: the exact critical-scale formula, two chirality conventions, and the event five-link

Grid programs that run low, a nesting of families, four branches at exact phases, and — numerically, in double precision — a strict local maximum among nearby unit five-links, on a neighbourhood of unestablished size



Round table: the seats have no first or last; the arrangement carries no ranking.

2026-10-21

Abstract

A recent preprint gives a computer-assisted proof that Wetzel's 30-60-90 triangle, shrunk by the factor 1.0048, contains a congruent copy of every arc of length one. How tight is that cover — which unit arcs need the most of it? A research programme searched for such arcs over fifteen rounds; this paper takes the first six, from grid linear programs to a stationary point among five-link polylines, and states each result at the strength an independent recomputation measured. We check the programme's exact formula for the least scale of a rotated polyline in the triangle, and show that the grid programs of its first rounds are relaxations that run low; we apply the programme's two conventions — rotations only, or rotations and reflections — to every candidate, where they move values by percents; we show that the five-link family nests in the seven-link one, so a reported seven-link deficit was the search's; we locate the four competing contact branches of the round's best five-link at exact phases — the method the programme's own audit proposed and its next round adopted; and we show, numerically and in double precision, that the root of the programme's nonsmooth KKT system, the event five-link with least scale 0.998903757132509, satisfies the second-order sufficient condition, so that it is a strict local maximum of the congruent least scale among all unit five-links near it, on a neighbourhood whose size is not established. No exact root is certified near the computed one. The paper proves no bound for Moser's worm problem, and lists the corrections found on the way.

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Status of this document. Preprint, version 0.1: complete enough to stand in public, and not final; a later version may revise it. Not prepared for journal submission. References are added before formal upload. Section 12 records who did which part.

0. What is claimed, and what is not

Every statement in this paper carries a status, given after it in italics. A statement with a proof carries a verdict on that proof. `PASS` and `PASS by cited theorem` are different verdicts and are never written as one: the first is a proof checked line by line — against the programme's text, or, for the few arguments this paper supplies itself, as written here — and the second a proof one of whose steps holds because a published theorem does, cited with its statement checked against its source. Every other statement carries one of six labels, and the labels are never merged:

label	meaning
`COMPUTED_HERE`	a figure recomputed by the recomputation arm with independently written code — symbolically where the text says *exact*, in floating point or at high precision where it says *numerical*
`DERIVED_FROM_VERIFIED_INPUTS`	follows from statements already verified here, with no step of its own beyond combining them
`CITED_THEOREM`	someone else's published theorem, stated here and not re-proved; its statement checked against its source
`INHERITED`	taken from the research programme's documents and not verified here; every use is marked
`UNMEASURED`	well posed, and not established here
`FAILED`	attempted and refused: false as stated, or true only in the corrected form given

A *numerical* figure is never a certificate. Every least scale over rotations in this paper is a minimum found by scanning the rotation and refining every local minimum; none is proved global. No interval

arithmetic is used in the runs this paper rests on, so its two main results, Theorems 6.1 and 7.2, are stated at a computed point and hold at the strength of double-precision figures.

Not claimed anywhere in this paper: any bound for Moser's worm problem — the least area of a convex set containing a congruent copy of every unit arc is open, and nothing here bears on it directly; that the triangle $T/1.0048$ covers every unit arc, which is a cited preprint and not checked here; that an exact root of the contact-event system exists near the computed one — that would need an interval Newton or Kantorovich argument, which neither the programme nor this paper makes; that any curve in this paper is the hardest unit arc for the triangle, or the best of its family — every search is a search; that the event five-link is the five-link family's maximum, or the only root of its system, or that its contact topology is the best one; the size of the neighbourhood on which it is a local maximum; anything about the 27 library curves the programme names without defining, or about the samples and candidates it did not log.

1. Introduction

1.1 The problem

A planar set C *covers* a curve if it contains a congruent copy of it. Moser asked for the convex set of least area that covers every curve of length one — every *unit arc* — and the question is open. It dates from 1966, by the account of Panraksa and Wichiramala [5] (we have not seen a primary source for the date); Norwood, Poole and Laidacker [4] give an early treatment of the problem. Partial answers come from explicit covers: a set shown to cover every unit arc bounds the least area from above.

One such cover is a 30-60-90 right triangle that Wetzel conjectured to cover every unit arc, as the preprint below reports; for his 30° sector of radius 1 the conjecture was proved for drapeable arcs by Movshovich and Wetzel [6] and in full by Panraksa and Wichiramala [5]. Wichiramala and Panraksa (preprint, arXiv:2606.14625) prove by a computer-assisted argument that Wetzel's triangle T covers every unit arc, reducing an uncovered arc to 599 second-order cone models each validated in interval arithmetic; since every certified lower endpoint exceeds 1.0048, the triangle $T/1.0048$ covers every unit arc too, by congruent copies, reflections allowed. *`CITED_THEOREM` (a preprint; its statement and convention checked against its text, nothing of its proof checked; RUN-002, RUN-003).*

For a unit arc γ, write $S(\gamma)$ for the least s such that a copy of γ moved by a rotation and a translation lies in $s \cdot T/1.0048$, and $S_{\mathrm{cong}}(\gamma) = \min\{S(\gamma), S(\bar{\gamma})\}$ for the least over both handednesses, $\bar{\gamma}$ being a mirror image. The cited theorem says $S_{\mathrm{cong}}(\gamma) \leq 1$ for every unit arc. The programme this paper recomputes asks the opposite question: how close to 1 can S_{cong} come — which unit arcs need the most of the cover?

1.2 What this paper supplies

The programme's first six rounds move from a library of curves under grid linear programs to a single five-link polyline at which four contact branches balance. This paper takes them in order and states what each established, at the strength an independent recomputation measured:

- §2 fixes the triangle, the programme's two conventions and its exact least-scale formula for a polyline;
- §3 shows that the grid programs of the first two rounds are relaxations and measures how far they run low;

- §4 shows that the five-link family nests in the seven-link family and both contain the symmetric three-links, so a reported shortfall of seven links was a search's, and explains the round's "most important structural finding" as a consequence of the imposed symmetry;
- §5 locates the four competing contact branches at exact phases, where each branch value is a closed form;
- §6 shows, numerically and in double precision, that the computed root of the programme's contact-event system satisfies the second-order sufficient condition for a maximum of the four-branch max-min, and transfers it to the curve's least scale on a neighbourhood of unestablished size;
- §7 extends that to every unit five-link near the root, chiral or not, at the same strength;
- §8 lists every result with its strength, what reproduced included, and §9 the corrections found on the way.

1.3 Where the mathematics comes from, and how it was checked

The rounds are those of a research programme on this question, rounds 01 to 06 of its fifteen; the programme's packages are public, byte for byte, on the AMRAL site (§10). Every statement below was checked by an independent recomputation (懸衡): every figure rebuilt with independently written code from the rounds' formulations, never by running the programme's code — which in these rounds the packages do not ship — every argument read, and every check drilled with planted faults. The checks are the runs RUN-002 to RUN-007 of the recomputation tree pinned in the front matter, one per round; each result below names the run that checked it. Where the paper reads a round's statement in a way the round does not spell out, it says so.

2. The triangle, the conventions and the exact formula

2.1 The triangle

Place $T/1.0048$ with its right angle at the origin and its legs on the positive axes:

$$T_0 = \{(x, y) : x \geq 0,\ y \geq 0,\ x/A + y/B \leq 1\}, \qquad A = \frac{3 + 4\sqrt{3}}{18 \cdot 1.0048}, \quad B = \frac{4 + \sqrt{3}}{6 \cdot 1.0048}.$$

Proposition 2.1. *$B/A = \sqrt{3}$, so T_0 is a 30-60-90 right triangle with its short leg on the x-axis; the square inscribed on its hypotenuse (for T undivided) has side exactly $1/3$, the definition of Wetzel's triangle in the cited preprint; and the outer normal of the hypotenuse points at 30°.* Indeed $3(4 + \sqrt{3}) = \sqrt{3}(3 + 4\sqrt{3})$. *`COMPUTED_HERE` (exact, symbolic; in floating point to 2.2×10^{-16}), RUN-002, RUN-006.* Of the two handednesses placed this way, only this one reproduces the first round's recorded scale for its adversarial candidate; the other gives a far larger value (RUN-002). Rotations φ are counterclockwise throughout.

2.2 Two conventions

A unit arc's critical scale depends on whether reflections are allowed. Under rotations and translations only it is $S(\gamma)$; under congruence it is $S_{\text{cong}}(\gamma) = \min\{S(\gamma), S(\bar{\gamma})\}$. The cited cover is by congruent copies, so a curve compared with it must be scored by S_{cong}. The programme's second round makes this distinction and keeps the two values apart from then on. It is not small: the first round's adversarial candidate has $S = 0.918731$ and $S_{\text{cong}} = 0.894545$; a re-optimized three-link of the second round

has $S = 1.007315039573$ — above the cover — and its mirror image fits at 0.9529389747. *COMPUTED_HERE (numerical; the minima over rotation found by scanning), RUN-002, RUN-003.* A mirror-symmetric curve has one value. In the third to fifth rounds every family searched is mirror-symmetric (§4, §5, §6); the sixth leaves it for the general five-link (§7).

2.3 The exact formula

Proposition 2.2 (exact least scale of a polyline). *Let γ be a polyline with vertices $q_0, \dots, q_n$, and let (x_i, y_i) be the vertices of γ rotated by φ. The least s for which a translate of the rotated polyline lies in sT_0 is*

$$s(\varphi) = \max_i \left(\frac{x_i}{A} + \frac{y_i}{B} \right) - \frac{\min_i x_i}{A} - \frac{\min_i y_i}{B}.$$

Proof. A polyline lies in a convex set exactly when its vertices do. The translate by t lies in sT_0 when $x_i + t_x \geq 0$, $y_i + t_y \geq 0$ and $(x_i + t_x)/A + (y_i + t_y)/B \leq s$ for all i. The third condition is easiest to meet with t_x, t_y as small as the first two allow, since $A, B > 0$: $t_x = -\min_i x_i$ and $t_y = -\min_i y_i$, which puts the leftmost vertex on the leg $x = 0$ and the lowest on the leg $y = 0$; the least s is then the formula. *Status:* PASS (the programme's second-round addendum gives it; checked against the three-normal linear program on random polylines to 10^{-10}, RUN-003). The critical scale is $S(\gamma) = \min_\varphi s(\varphi)$.

Each of the three terms is a maximum or minimum of finitely many sinusoids in φ, so s is continuous and piecewise smooth, with a kink wherever the active vertex of a term changes. The minima of s over φ that matter in this paper all sit at such kinks (§5), which is why a smooth one-dimensional minimizer stops beside them.

3. Grid programs run low

3.1 The relaxation

The first round places a curve by linear programming over a grid of 192 support directions θ_j: for a fixed rotation it finds the least s, with a translation, such that every support value of the curve is at most s times the container's in each grid direction. On a finite grid this asks the curve to fit the polygon cut out by the grid's support lines of sC, which contains sC.

Proposition 3.1. *The grid program's value never exceeds the curve's true scale in the container; it equals it for the triangle when the triangle's three outer normals are grid directions. Status:* PASS (RUN-002, RUN-003).

The 192-direction grid of the first round holds all three of T_0's outer normals, so its triangle values are exact up to the search. The second round moved to 256 equally spaced directions, which miss the hypotenuse's normal: its nearest grid directions give support lines through the triangle's acute vertices, the polygon they cut leaves a sliver beyond the hypotenuse, and every value on it is a relaxation value. *COMPUTED_HERE, RUN-003.*

3.2 What the relaxation did to the first two rounds

- **The disc and the sector.** On the 192-direction grid the unit segment in the disc of radius 1/2 comes out at exactly $\cos(\pi/192) = 0.99986614$ — by the relaxation's geometry, the grid polygon's widest chord being $1/\cos(\pi/192)$ times the disc's; the program's value agrees with it to rounding — the value the first round printed as the sector's library maximum; the disc's printed maximum, 0.99986613, is the same value within that record's logged residual. Both containers

cover every unit arc — the disc because each point of a unit arc lies within 1/2 of its midpoint, Wetzel's 30° sector by Panraksa and Wichiramala [5] — so neither value can exceed 1. The true maximum in both is 1, the unit segment's, by the same argument (the sector's value computed on a dense grid is 0.99999993); the other records in the two containers, 27 of whose curves are inherited, are far below 1 on the grid, and stay below 1 when divided by $\cos(\pi/192)$. *COMPUTED_HERE (by argument, with the program's values to rounding), RUN-002.*

- **The second round's numbers.** Its main report's three-link value 0.9933108210 and its U-shaped curve's 0.9512368779 are values on the 256-direction relaxation. On the exact formula of Proposition 2.2, with the minima over rotation found numerically, the grid candidate is 1.0008373696 — orientation-preserving, above the cover — and its mirror image fits at 0.954897, so its congruent value is below 1, which the round did not compute; U is 0.957829839129. The round's own exact addendum carries the first and the last of these corrections. The main report's conclusion that the three-link family "尚未逼近 Wetzel 認證尺度 1" (has not yet approached the certified scale 1) does not hold as stated: under the congruent convention the round prescribes, its own corrected curve reaches 0.998574, and the recomputation's search of the family 0.998698. *COMPUTED_HERE (numerical), RUN-003; the conclusion FAILED.*

3.3 Records above the program's own minimum

Eleven of the 21 records of the first round that can be rebuilt stop above the scanned minimum of the same grid program — which the recomputation reads as a local search stopping early — nine of them by 5.3×10^{-7} to 4.4×10^{-6} (the largest, V_60 in the sector) and two by less; none lies below it by more than its own logged residual. The triangle's headline U-shaped curve is recorded at 0.95783224 against a minimum of 0.95782984, so its last printed digits are not the curve's; its adversarial candidate is recorded at 0.9187329185, 1.6×10^{-6} above the minimum of its own handedness, 0.918731. The second round's stated values are likewise 2.9×10^{-9} and 7.2×10^{-9} above the minima of its own program. *COMPUTED_HERE (numerical; the minima found by scanning, not proved global), RUN-002, RUN-003.*

4. Three-, five- and seven-link families

4.1 The families

The second round's **three-link** family has directions $(\alpha_1, 0, -\alpha_3)$ and lengths summing to 1; it is chiral in general, and the example of §2.2 is one of its members. Its **symmetric** members, with directions $(\alpha, 0, -\alpha)$ and lengths $(l, 1-2l, l)$, are mirror-symmetric. The third round searches two mirror-symmetric families: **five links** with directions $(\alpha, \beta, 0, -\beta, -\alpha)$ and lengths $(l_1, l_2, l_3, l_2, l_1)$, and **seven links** with $(\alpha, \beta, \gamma, 0, -\gamma, -\beta, -\alpha)$ and $(l_1, l_2, l_3, l_4, l_3, l_2, l_1)$; every length positive, the total 1. A mirror-symmetric curve is congruent to its own reflection, so its two handednesses agree and the question of §2.2 does not arise inside these two families. *PASS, RUN-004.*

4.2 Nesting

Proposition 4.1. *Every five-link curve is a seven-link curve, and every symmetric three-link — directions $(\alpha, 0, -\alpha)$, lengths $(l, 1-2l, l)$ — is both a five-link and a seven-link curve. Hence the seven-link family's supremum of the least scale is at least the five-link family's, and both are at least the symmetric three-links'.*

Proof. Set $\gamma = \beta$ and take the seven-link lengths $(l_1, l_2/2, l_2/2, l_3)$: the second and third links point the same way, so the curve is the five-link $(\alpha, \beta, 0, -\beta, -\alpha)$, $(l_1, l_2, l_3, l_2, l_1)$, with its second side marked at the midpoint. The symmetric three-link with lengths $(l, 1-2l, l)$ is the five-link with $\beta = \alpha$ and lengths $(l/2, l/2, 1-2l)$, and so a seven-link too. If a search keeps its angles strictly ordered, $\gamma = \beta$ lies on the boundary of its parameter set, and the supremum is still at least the five-link's, since the least scale over rotations depends continuously on the curve. *Status:* PASS (RUN-004, which also writes the round's own five-link as a seven-link and finds its value unchanged, 0.998754371668).

The third round reports a best five-link at 0.998754371668 and a best seven-link at 0.998674157956. Its README and the boxed conclusion of its §6 attribute the shortfall to the added links, which "大量退化為近平行子段" (largely degenerate into near-parallel sub-segments), and its §12 calls it the more important limitation and draws a lesson from it; its §5 states the shortfall as a fact about its search, and its §11 allows that "七段低於五段可能是族本身、參數化或搜尋不足造成" (seven below five may be due to the family itself, the parameterization or insufficient search). Proposition 4.1 rules out the family itself: the seven-link value is where the search stopped. And degeneration cannot lower a family's supremum — a seven-link whose added links are exactly parallel is a five-link, with that curve's value — though near-parallel pieces can make a search stall. *The README's and §6's conclusion FAILED, RUN-004.* The reported seven-link is also 2.36×10^{-5} below the equal-thirds symmetric three-link, $\alpha = 82.435472°$, all lengths 1/3, least scale 0.998697791610, which both families contain. *COMPUTED_HERE (numerical), RUN-003, RUN-004.*

The families go higher than the round reported. The recomputation's own searches — Nelder–Mead from two starts per family, each candidate's least scale taken on a rotation scan with every local minimum refined, the winner re-evaluated on 65,536 rotations — reach 0.998903093 with five links and 0.998904534 with seven, the second 1.44×10^{-6} above the first. *COMPUTED_HERE (numerical; searches, not proved maximal), RUN-004.* Against the equal-thirds three-link the recomputation's search found, the round's five-link gains 5.66×10^{-5}, not the 1.80×10^{-4} it states against an orientation-preserving baseline; the round's reading that three links are not the end point of its search still holds, by less. *COMPUTED_HERE, RUN-004.*

4.3 The double contact on the hypotenuse is the symmetry's

The third round singles out a structural feature of its best curves: both endpoints on the hypotenuse at once, "這是第 3 輪最重要的結構發現" (this is round 3's most important structural finding).

Proposition 4.2. *Let γ be symmetric about a line ℓ, and let φ be the rotation that makes ℓ parallel to an outer normal n of T_0. Then every pair of mirror-image vertices of the rotated curve has equal values of $n \cdot x$.* Two points that are mirror images across a line parallel to n differ by a vector orthogonal to n. *Status:* PASS (RUN-004).

For the symmetric families, $\varphi = 2\pi/3$ makes the axis parallel to the hypotenuse's normal, and the two endpoints tie on the hypotenuse by Proposition 4.2. Both of the round's curves reach their least scale at $\varphi = 2\pi/3$, to the last bit of a double, at a kink between the two endpoints' branches, with one-sided slopes -0.269 and $+0.615$ (five links) or $+0.616$ (seven links). *COMPUTED_HERE (numerical; the tie exact, by symmetry), RUN-004.* So the double hypotenuse contact is what the imposed symmetry gives at the rotation where the minimum falls; what the search found is that the minimum falls there, with one interior vertex on each leg. *The contact COMPUTED_HERE; the round's reading FAILED in the sense of §0 — true only in the corrected form given: the search found the phase, and the symmetry gives the contact (RUN-004 §4).* The same argument at $\varphi = 3\pi/2$, where the axis is parallel to the leg $x = 0$'s normal, ties the endpoints on that leg (§5).

5. Four branches at exact phases

5.1 The round's four branches

The fourth round starts from the third round's five-link, tracks the four lowest local minima of $s(\varphi)$ over rotation as functions $m_r(\eta)$ of four parameters η — which the recomputation reads as $\eta = (l_1, l_2, \beta, \delta)$ with $\delta = \alpha - \beta$ and $l_3 = 1 - 2l_1 - 2l_2$, the reading under which the round's sensitivity matrix reproduces to its third or fourth decimal (§9.1) — and levels them by solving

$$\max z \quad \text{subject to} \quad m_r(\eta) \geq z, \quad r = 1, \dots, 4.$$

It reports a least scale of 0.998903750476, reached at $\varphi = 3\pi/2$, and its headline holds: levelling the weakest branches raises the five-link's least scale without adding links. *COMPUTED_HERE (numerical, the least scale to* 5×10^{-15}*), RUN-005.* It also reports the four tracked values level to $2.734812376559\times 10^{-12}$. That is the tracker's: its one-dimensional minimizer stopped beside each branch's kink, and the four true branch minima span 1.30×10^{-8} — as the round's own cusp audit concedes, withdrawing the claim. *FAILED as stated, RUN-005.*

5.2 Where the branches are

The round's cusp audit lists, among the remedies for the next round, "精確特殊相位" (exact special phases), and the fifth round writes the four branch phases from the active contact identities (§6.1). At the fourth round's own curve the recomputation finds the four minima there, to 2.2×10^{-15}:

Proposition 5.1. *At the fourth round's curve the four branch minima lie at exact phases where the contact identity changes:*

branch	contact change	phase
B1	link 1 parallel to the leg $x = 0$: vertices 0 and 1 tie on it	$\pi/2 - \alpha$
B2	link 2 parallel to the leg $x = 0$: vertices 1 and 2 tie on it	$\pi/2 - \beta$
B3	the symmetry axis parallel to the hypotenuse's normal: the endpoints tie on it	$2\pi/3$
B4	the symmetry axis parallel to the leg $x = 0$'s normal: the endpoints tie on it	$3\pi/2$

The identities are exact (a link parallel to a leg; a tie by Proposition 4.2); that the minima lie there is COMPUTED_HERE (numerical: the recomputed minima at these phases to 2.2×10^{-15}*), RUN-005.*

Each branch value is then the exact scale at a known phase, $m_r(\eta) = s(\varphi_r(\eta); \eta)$ — a closed form, smooth in η wherever the contact identities hold. The non-smoothness the audit found is in φ, not in η, so its further conclusion — that the gradients $\nabla_\eta m_r$ alone will not do — does not follow at this point: there they are well defined, and computed from the closed forms they agree with central differences of the true minima to 1.5×10^{-9}. *COMPUTED_HERE, RUN-005.*

Solved with these closed forms, the round's own levelling problem brings the four branches level to 1.1×10^{-15} at least scale 0.998903757133, 6.7×10^{-9} above the round's value; a full rotation scan gives

the same least scale. The four gradients there have rank 3 with a positive least-norm multiplier on the simplex, residual 6.5×10^{-8}, and first- and second-order probes find no direction that raises the lowest branch. *COMPUTED_HERE (numerical; the solver ended with status 8, so the point was examined directly), RUN-005.* At the round's own point, by contrast, only B4 is active; on the recomputed gradients the round's pressure vector leaves a residual of 2.0×10^{-4}, and even the least-norm combination on the simplex leaves 8.3×10^{-6}: read strictly, some direction raises all four branches at once there. The round's reading that no direction does holds at the levelled point, to 10^{-7}. *COMPUTED_HERE, RUN-005.* The levelled point is, to 10^{-15} in value, the fifth round's root, reached by another route.

6. The event five-link

6.1 The contact-event system

The fifth round writes the four branch values as closed forms in $p = (l_1, l_2, \beta, \delta)$, with $\alpha = \beta + \delta$, $l_3 = 1 - 2(l_1 + l_2)$, vertices $q_0 = 0$, $q_1 = l_1 v_\alpha$, $q_2 = q_1 + l_2 v_\beta$, $q_3 = q_2 + l_3(1, 0)$, at the phases $\pi/2-\alpha, \pi/2-\beta, 2\pi/3$ and $3\pi/2$, and observes that $m_1 = m_2 = m_3 = m_4$ alone gives three equations in four unknowns — "一般形成一維候選族" (in general a one-parameter family of candidates). It adds branch pressures $\mu_r \geq 0$ with $\sum_r \mu_r = 1$ and $\sum_r \mu_r \nabla_p m_r = 0$, which makes nine equations in the nine unknowns $(p, s, \mu_1, \dots, \mu_4)$: the first-order conditions of the levelling problem of §5.1. *The round's four closed forms are the right ones: they equal the exact scale of Proposition 2.2 at the event phases to* 3.3×10^{-16}*; COMPUTED_HERE, RUN-006.*

A starting point. The recomputation's levelled point of §5.2 is

$$p = (0.19431311717532038,\ 0.14159307841073046,\ 1.4082767269127132,\ 0.04759060098538986)$$

(lengths, and β, δ in radians); it differs from the round's root by at most 6.0×10^{-9} in each parameter. Newton's method on the nine equations, started there with equal pressures, reaches the round's root to 3.1×10^{-15}; and from 40 starts of its own, each unknown of the round's root perturbed by a Gaussian of standard deviation 10^{-2}, it returns every time, and at 3×10^{-2} in 32 of 40. *COMPUTED_HERE, RUN-005, RUN-006.* At the root the nine equations hold to 5.7×10^{-16} and the four branch values are level at

$$s_{\text{event}} = 0.998903757132509,$$

the least scale over all rotations, reached at all four event phases; the next local minimum over rotation is 0.030 higher. The pressures are all positive; at the levelled point of §5.2 the least-norm multiplier on the simplex is $(0.0280, 0.0287, 0.7365, 0.2067)$, the largest on the $2\pi/3$ branch. *COMPUTED_HERE (numerical, double precision; the least over rotation by scanning), RUN-006.* Call this curve the **event five-link**.

6.2 The second-order condition

The round calls its root a numerically isolated stationary point. At the computed root more holds.

Theorem 6.1 (numerical). *At the computed root — where the nine equations hold to* 5.7×10^{-16}*, all four pressures are positive, the branch-gradient matrix G has rank 3 to double precision and the Lagrangian's curvature along its null vector is* -0.1227 *— the second-order sufficient condition for a strict local maximum of the four-branch max-min* $\min_r m_r(p)$ *holds. An exact root near the computed one at which these hold is a strict local maximum of the model.*

Proof. The problem is: maximize z over (p, z) subject to $m_r(p) \geq z$, with Lagrangian $L = z + \sum_r \mu_r(m_r(p) - z)$. Let G be the 4×4 matrix whose rows are the branch gradients. With every μ_r positive, all four constraints are strongly active, and the condition asks that $d^{\mathsf{T}}(\sum_r \mu_r \nabla^2 m_r)d < 0$ for every nonzero (d, d_z) in the critical cone $\{(d, d_z) : Gd = d_z\mathbf{1}\}$. Stationarity is $\mu^{\mathsf{T}}G = 0$, so G has rank at most 3; its singular values are 5.68, 0.314, 0.0407 and 1.7×10^{-17}. A critical direction has $d_z = d_z(\mu^{\mathsf{T}}\mathbf{1}) = \mu^{\mathsf{T}}Gd = 0$, so $Gd = 0$, and with rank 3 the cone is the line through G's unit right null vector v. Along it

$$v^{\mathsf{T}}\Big(\sum_r \mu_r \nabla^2 m_r\Big)v = -0.1227,$$

negative. *The argument PASS; the singular values and the curvature COMPUTED_HERE (numerical, double precision: central differences of complex-step gradients, and again from the closed forms' values), RUN-006; the existence of an exact root UNMEASURED.*

The same line has a plainer meaning. Since G has rank 3 and $\mu^{\mathsf{T}}\mathbf{1} = 1 \neq 0$, the differences $m_r - m_4$ have independent gradients, so the level set $m_1 = m_2 = m_3 = m_4$ is a smooth curve through the root — the round's one-parameter family — with tangent v; along it the common value has zero first derivative and, at unit speed, second derivative -0.1227. Probes agree: along the straight line $p \pm tv$ the least closed form falls by 2.04×10^{-9} at $t = 10^{-4}$, and along the level curve, corrected back onto it by Newton's method, the common value falls quadratically too. *COMPUTED_HERE, RUN-006.*

The round's isolation rests on the same scalar. The KKT Jacobian is nonsingular exactly when the constraint gradients in (p, z), the rows of $[G, -\mathbf{1}]$, have full rank 4 — they do, because $\mu^{\mathsf{T}}\mathbf{1} = 1$ — and $v^{\mathsf{T}}(\sum \mu_r \nabla^2 m_r)v \neq 0$. The round reports rank 9; this paper adds the sign.

6.3 From the model to the curve

Corollary 6.2 (numerical). *If Theorem 6.1 holds at an exact root, that root is a strict local maximum of the least scale $S(p) = \min_\varphi s(\varphi; p)$ over the symmetric five-link family, on some neighbourhood whose size is not established.*

Proof. At the computed root every contact identity holds with a strict margin: in each event's ledger the terms not tied have a single active vertex and the tied term exactly two, each at least 0.0067 from the next vertex; the one-sided derivatives of s at the four events are strictly signed, $|D_\pm| \geq 0.0238$; and every other local minimum over rotation is at least 0.030 higher. All of these move continuously with p, so on some neighbourhood each branch's minimum stays a kink at its event phase with the same identities, and no other phase comes lower: there $S(p) = \min_r m_r(p)$, and Theorem 6.1 applies. *The argument PASS; its inputs COMPUTED_HERE (numerical, double-precision margins), RUN-006; the neighbourhood's size UNMEASURED.*

At every line probe, and at sixty random points around the root at each of two scales, the least scale over all rotations equals the least closed form to 5.6×10^{-16}. *COMPUTED_HERE, RUN-006.* How large the neighbourhood is, and whether an exact root lies within it, would need an interval certificate, which the round plans and this paper does not attempt.

7. Among all nearby five-links

The sixth round leaves the symmetric family for the general five-link near the root: four antisymmetric parameters $d = (d_{l_1}, d_{l_2}, d_\alpha, d_\beta)$ join the four symmetric ones, giving lengths $(l_1 + d_{l_1}, l_2 + d_{l_2}, l_3, l_2 - d_{l_2}, l_1 - d_{l_1})$ and directions $(\alpha + d_\alpha, \beta + d_\beta, 0, -\beta + d_\beta, -\alpha + d_\alpha)$, and every curve is scored by S_{cong}.

Proposition 7.1. *(i) The eight parameters are a chart of every unit five-link near the root, up to rigid motion. (ii)* $S(\bar{\gamma}_{p,d}) = S(\gamma_{p,-d})$*: the mirror image of the* d*-curve is the* $-d$ *curve, up to rigid motion.* For (i), the map from (p, d) to the five lengths and four free directions, with the middle link horizontal and its length fixed by the total, is linear and invertible ($l_1 = (L_1+L_5)/2$, $d_{l_1} = (L_1-L_5)/2$, $\alpha = (\theta_1-\theta_5)/2$, $d_\alpha = (\theta_1 + \theta_5)/2$, and so on). For (ii), reflect in the x-axis, traverse backwards and turn by π. *Status:* PASS; (ii) also checked numerically at random points to 4.4×10^{-16}, RUN-007.

The four contact identities stay meaningful for the general five-link: link 1, then link 2, parallel to the leg $x = 0$, at $\varphi = \pi/2 - (\alpha + d_\alpha)$ and $\pi/2 - (\beta + d_\beta)$; the endpoints tied on the hypotenuse, at $\varphi = 2\pi/3 - \psi$; and on the leg $x = 0$, at $\varphi = 3\pi/2 - \psi$, where ψ is the direction of the chord from the near endpoint to the far one (for the symmetric curve, 0). At these phases each branch value is a closed form $m_r(p, d)$ in all eight parameters, and near the root

$$S_{\text{cong}}(p, d) = \min_r \min\{m_r(p, d),\ m_r(p, -d)\}.$$

At the root the closed forms are the event scale to 3.3×10^{-16}*; sampled near it, the congruent least scale equals this minimum to* 1.1×10^{-15}*;* COMPUTED_HERE*, RUN-007.*

Theorem 7.2 (numerical). *At the computed root, the eight-parameter model satisfies the second-order sufficient condition for a strict local maximum. If this holds at an exact root, that root is a strict local maximum of the congruent least scale among all unit five-links near it, up to rigid motion, on a neighbourhood whose size is not established.*

Proof. Maximize z subject to $m_r(p, d) \geq z$ and $m_r(p, -d) \geq z$. At $d = 0$ the two sets of constraints share their symmetric gradients G_p and have opposite antisymmetric gradients $\pm G_d$. The 4×4 matrix G_d is nonsingular — its singular values are 2.52, 0.239, 0.135 and 0.0394 — so stationarity in d forces equal multipliers $\mu/2$ on the two sets. A critical direction (d_p, d_d, d_z) satisfies $G_p d_p + G_d d_d = d_z \mathbf{1}$ and $G_p d_p - G_d d_d = d_z \mathbf{1}$, so $G_d d_d = 0$, hence $d_d = 0$, and $G_p d_p = d_z \mathbf{1}$, hence $d_z = 0$ and d_p is along v, as in Theorem 6.1. The critical cone is the line $(v, 0)$, and along it the Lagrangian's curvature is the symmetric problem's, -0.1227. The margins of Corollary 6.2 hold for the curve and its mirror image alike, so the congruent least scale equals the model on some neighbourhood. *The argument* PASS*;* G_d*'s singular values and the curvature* COMPUTED_HERE *(numerical, double precision), RUN-007; the neighbourhood's size and the exact root* UNMEASURED*.*

This is what the round's evidence was gathered to suggest — its interpretation note's "目前資料支持五連桿事件根對小型手 (the present data support the event root's local stability against small chirality breaking) — at the strength of a double-precision second-order condition rather than of sampling, and stronger: on that neighbourhood no unit five-link, chiral or not, does better. The round's four single-mode scans all peak at zero, and on the true least scales they do so strictly; but for one antisymmetric mode $S_{\text{cong}}(t) = \min\{S(t), S(-t)\}$ is even in t by Proposition 7.1(ii), so a peak at zero is forced unless every branch is stationary in that mode, and four peaks along four axes could not exclude a gain along a combination — which is what G_d's nonsingularity excludes. *COMPUTED_HERE (numerical), RUN-007.* One reading does not hold: the interpretation note's statement that the eight-dimensional search "重新收斂到近鏡像對稱" (re-converged to near mirror symmetry). Its best candidate lies 3.47×10^{-4} below the root, a member of the family searched, with antisymmetric lengths 5.6 % and 8.1 % of the symmetric ones. *COMPUTED_HERE; the reading* FAILED*, RUN-007.*

8. Every result, and the strength at which it holds

result	strength	run
$T/1.0048$ covers every unit arc, by congruent copies	CITED_THEOREM (a preprint, arXiv:2606.14625; its proof not checked)	RUN-002, RUN-003
Proposition 2.1: the triangle's geometry	COMPUTED_HERE, exact (symbolic)	RUN-002, RUN-006
the two conventions applied to every candidate	COMPUTED_HERE, numerical	RUN-002, RUN-003
Proposition 2.2: the exact formula (the programme's)	PASS	RUN-003
Proposition 3.1: grid programs are relaxations	PASS	RUN-002, RUN-003
reproduced, round 01: the candidate's recorded translation, tight directions, 98 % test (orientation-preserving) and whole skew field; the triangle's areas	COMPUTED_HERE, numerical (the skew field to rounding)	RUN-002
reproduced, round 02: the contact-pressure ledger to the digit; every logged scan row of the curves it defines; every exact value of its two addenda, to 10^{-10}	COMPUTED_HERE, numerical	RUN-003
reproduced, round 03: its three critical scales to 1.4×10^{-13}, its resolution table, contact ledgers and metrics blocks	COMPUTED_HERE, numerical	RUN-004
reproduced, round 04: its least scale, nine local minima, contact ledgers and pressure arithmetic; its headline	COMPUTED_HERE, numerical	RUN-005
reproduced, round 05: its closed forms, its nine equations at its root, its Jacobian's rank 9, its 40 trials	COMPUTED_HERE, numerical	RUN-006
reproduced, round 06: its reference, mode profiles, escape curve, and logged census and audit rows	COMPUTED_HERE, numerical	RUN-007
the library maxima, the second round's relaxation values, the records against scanned minima	COMPUTED_HERE, numerical	RUN-002, RUN-003
Proposition 4.1: nesting of the families	PASS	RUN-004

result	strength	run
the recomputation's search values (three-, five- and seven-link)	COMPUTED_HERE, numerical; not proved maximal	RUN-003, RUN-004
Proposition 4.2: mirror vertices tie at the symmetry phases	PASS	RUN-004
Proposition 5.1: the four branch minima at exact phases	identities exact; positions COMPUTED_HERE, numerical	RUN-005
the levelled point at 0.998903757133	COMPUTED_HERE, numerical	RUN-005
the event five-link, $s_{\text{event}} = 0.998903757132509$	COMPUTED_HERE, numerical, double precision	RUN-006
Theorem 6.1: second-order sufficient condition at the computed root	argument PASS; inputs COMPUTED_HERE, numerical; exact root UNMEASURED	RUN-006
Corollary 6.2: strict local maximum among symmetric five-links	argument PASS; margins numerical; neighbourhood UNMEASURED	RUN-006
Proposition 7.1: the chart and the mirror identity	PASS	RUN-007
Theorem 7.2: strict local maximum among all nearby unit five-links	argument PASS; inputs numerical; neighbourhood and exact root UNMEASURED	RUN-007
global optimality of any curve here; uniqueness of the root; optimality of the contact topology	UNMEASURED	—
the 27 undefined library curves, the unlogged samples and candidates	INHERITED, unchecked	RUN-002, RUN-003, RUN-007

9. Corrections found on the way

9.1 Statements and figures that do not hold as written

A selection; the inventory of the six reports holds 30 correction rows, and the reports hold every one, save one consequence in the third row, derived here from two of RUN-002's values and stated in no report. Rows the programme itself corrected, in an addendum or an audit of the same round, say so.

round	the programme says	the recomputation finds	run
01	the disc's and the sector's library maxima 0.99986613 and 0.99986614	the 192-direction grid's $\cos(\pi/192)$; the true maximum is 1	RUN-002

round	the programme says	the recomputation finds	run
01	the triangle's headline U-shaped curve at 0.95783224, and its other records	nine of the 21 rebuildable records lie above the program's scanned minimum by 5.3×10^{-7} to 4.4×10^{-6}; the U's minimum is 0.95782984	RUN-002
01	the adversarial candidate's critical scale 0.9187329185, compared with the certified cover	1.6×10^{-6} above the minimum of its handedness, 0.918731; and an orientation-preserving value, whereas the cover is by congruent copies, under which the candidate fits at 0.894545 — so the round's 98 % test, a positive defect left at 98 % of the recorded scale, holds for orientation-preserving placements only: the mirror image fits below that scale (`DERIVED_FROM_VERIFIED_INPUTS`, from the two values)	RUN-002
01	35 curves tested	the records hold 34	RUN-002
02	the three-link family "has not yet approached" the certified scale 1 (U 0.9512368779, best 0.9933108210)	relaxation values of a grid that misses the hypotenuse's normal (the round's own exact addendum corrects them); its corrected curve reaches 0.998574	RUN-003

round	the programme says	the recomputation finds	run
02	the addendum's body: the 1.007315 three-link does not challenge the certificate "because the result is still below 1"	the figure is above 1; the round's own chirality correction puts its mirror image at 0.952938974740 (reproduced), consistent with the cover; only the body was not updated	RUN-003
03	seven links do not improve on five; the added links degenerate (README, §6; the lesson of §12)	the seven-link family contains every five-link; the shortfall is the search's	RUN-004
03	the five-link gains 1.80×10^{-4} over three links	5.66×10^{-5} over the equal-thirds three-link the recomputation's search found	RUN-004
03	the summary's metrics blocks give each curve's minimum	an unrefined grid's, 1.37×10^{-4} to 1.38×10^{-4} above the least scale; and the contact-pressure entropy 0.963370649827 belongs to a scaling of the formula's rows (with unit normals, 0.965163253185), not to any curve	RUN-004
04	the four branches level to $2.734812376559 \times 10^{-12}$	the tracker's values; the true minima span 1.30×10^{-8} (the round's own cusp audit concedes it)	RUN-005
04	the sensitivity matrix to eight decimals	right to its third or fourth decimal	RUN-005

round	the programme says	the recomputation finds	run
04	no direction raises all four branches, at the round's point	only B4 is active there; the round's pressure vector leaves 2.0×10^{-4} on the recomputed gradients, and no combination on the simplex balances them (least residual 8.3×10^{-6}); the reading holds at the levelled point	RUN-005
05	the one-sided derivatives to nine decimals; $\sigma_{\min} = 3.607219048175 \times 10^{-2}$, $\kappa = 162.781733366$	one-sided differences, right to within 4×10^{-8}; a difference Jacobian's, the exact ones $3.607219010513 \times 10^{-2}$ and 162.781735059	RUN-006
06	the modes' largest chirality gaps 1.686×10^{-3}, 2.448×10^{-3}, 1.329×10^{-2}, 5.370×10^{-3}	differences of grid samples; on the least scales 1.743×10^{-3}, 2.083×10^{-3}, 1.330×10^{-2}, 5.069×10^{-3}	RUN-007
06	the eight-dimensional search re-converged to near mirror symmetry	its best lies 3.47×10^{-4} below the root, with antisymmetric lengths 5.6 % and 8.1 % of the symmetric ones	RUN-007

9.2 What could not be checked, or stays open

- The first round names 27 library curves without defining them; their records are the round's. Its search history's best value by the last generation, 0.93451, does not match the candidate it reports, 0.91873; whether the history and the final evaluation measure differently, or the reported candidate is not the best the search found, the package does not say. *`INHERITED, RUN-002.`*
- The second round's two scanned arcs, the interior of its U-family surface and its jump counts follow no stated rule that reproduces them, and its asymmetric topology-guided candidates are not logged. *`INHERITED, RUN-003.`*
- The fourth round's sensitivity rows differ from the branches' derivatives for a reason the package, which ships no code, does not show. *`UNMEASURED, RUN-005.`*
- The sixth round's 1600 census samples and 6300 audit samples are not logged, and its audit

radii are in units the documents do not state; its logged census and audit rows reproduce. *INHERITED, RUN-007.*

- The packages' figures (images) were not examined. *UNMEASURED, RUN-004 to RUN-007.*

10. Reproduction and attachments

What a reader can check without the archive. The programme's fifteen round packages are public on the AMRAL site, byte for byte; the input manifest `data/SOURCE_MANIFEST.v1.json` gives each package's address, size and SHA-256 and the SHA-256 of each of its 282 members, and every run refuses a package that does not match. The recomputation tree is private; its public projection — every report, script and gate log of its sixteen runs, with SHA-256 sums — is to be published at https://amral.evemisslab.com/moser/recompute/ (English: https://amral.evemisslab.com/en/moser/recompute/), one page per round, with this line's second-layer pages, before this paper is released. Every load-bearing number in this paper occurs in the gate logs of RUN-002 to RUN-007, in their citation snapshots `data/external/round0[12]-citations.json`, or in `data/paper-figures.json` (Appendix A), which adds up the drill records, takes the differences of logged fields that §3.3 and §4.2 rest on, and quotes the programme's own figures with the line they stand on; the paper's front matter declares these files as its sources, and the standard's checker traces every figure against them alone.

The instrument can fail, and was made to. Every check of every run was drilled: faults were planted in its inputs, and each had to be caught by exactly the checks named for it, while controls — the unmodified inputs and harmless rewrites — had to stay green. Over the six runs of this paper: 111 planted faults, all 111 caught by the named check and none uncaught or caught by another check; 24 controls, none disturbed; 55 checks; 999 must-stay-green assertions held. Over the whole recomputation tree, of which these runs are part: 352 planted faults, 352 caught by the named check, 64 controls undisturbed, 139 checks.

Commands. Python 3.14 [7] with mpmath 1.3 [8], numpy 2.4 [9], scipy 1.17 [10] (its HiGHS solver [12]) and sympy 1.14 [11]; no network beyond fetching the packages. From the line's directory (`moser-verification-xuanheng/` at the pinned commit), in this order — RUN-005 reads RUN-004's committed gate log, and RUN-006 reads RUN-005's:

```
python code/fetch_source.py
python code/run002_round01_support_skew_lp.py --drill
python code/run003_round02_phase_scan_chirality.py --drill
python code/run004_round03_symmetric_links.py --drill
python code/run005_round04_branch_equalization.py --drill
python code/run006_round05_contact_events.py --drill
python code/run007_round06_chirality_escape.py --drill
```

Each `--drill` run checks the package, drills every check and rewrites its gate log under `data/gate-logs/`; a re-run is correct when that log is byte-identical to the published one. RUN-002 takes under an hour on one core. `python code/paper_figures.py` rewrites `data/paper-figures.json` and `python code/source_manifest.py` the input manifest, on the same terms. The paper is typeset by the shared build tool of the crystallization repository, which refuses to write a PDF from which a character would be dropped.

11. What comes next in the line

The event five-link is not the end of the search. The eighth round's recomputation (RUN-009, outside this paper's evidence) finds, numerically, that a finite-width curvature layer at the event five-link's own parameters raises the least scale, so that in Hausdorff distance it is not a local maximum among curves; the programme's later rounds climb from there. That ascent, and the interval certificates found for some of its steps, are the subject of a companion paper; how the least scale of such a curve is certified over the whole rotation circle is the subject of another. This paper stops where the polygonal stage does.

12. Contributions

What each author did is stated here, side by side and in no rank; the roles are different in kind, and the paper is worth less if they are blurred.

Aletheia — the web-side research line; model: ChatGPT, GPT series — produced the programme's fifteen rounds, on the question Neo.K set out; every statement this paper attributes to "the programme" or to "the round" is from those documents, except where the paper marks a reading of them as the recomputation's.

Neo.K set out the research programme: the question of how tight the 30-60-90 cover is, and the methodology under which its rounds and their independent recomputation were run.

懸衡 (Xuán Héng) — the independent recomputation; model: Claude, Opus series — did the six runs this paper names and wrote this paper: every `COMPUTED_HERE` figure rebuilt with independently written code; every argument read against its source; the drills of §10; and the additions marked as the recomputation's where they occur — the measurement of the first round's grid values in §3 (the second round's own addendum caught its grid's), the nesting argument of Proposition 4.1 and the symmetry tie of Proposition 4.2, the families' higher search values, the grid candidate's mirror value, the location of the four branch minima at the fourth round's own curve and the levelling there, the second-order argument of Theorem 6.1 and its transfer in Corollary 6.2, the eight-parameter argument of Theorem 7.2, and the corrections of §9 that the programme did not make itself.

Readings. Before each of the six reports was committed, a separate instance of the same model (model: Claude Opus, version 5.5, run as a read-only subagent of 懸衡's Claude Code session), briefed only with the report and its evidence, read it independently; its findings were applied, and the reports record them. Two more such readings read this paper before it went to main, and their findings were applied. The paper was then reviewed by 墜衡 (Zhuì Héng), the lab's proof-side verification arm, who recomputed the event five-link's four branches, the singular values of G, the pressures and the Lagrangian's curvature along the null direction from the definitions of §2 and §6 with code of their own, checked the arguments of Theorems 6.1 and 7.2 — the critical cone and the constraint qualification included — and found that the first round's 98 % test does not hold under the congruent convention, which §8 and §9.1 now say.

Priority, stated plainly. The question, the families, the exact formula, the two conventions, the exact phases as a method and the event system are the programme's. What is offered as this line's own is narrower: an independent check of every step, the strength at which each step stands, the arguments listed above, and the corrections. The covering theorem it relies on is Wichiramala and Panraksa's.

Acknowledgements

We thank Anthropic, whose Claude models the lab's verification and recomputation lines run on, and OpenAI, whose ChatGPT models the programme's documents were written with. Not every AI model or instance that took part in this work is named in it — among them instances that the named authors ran as helpers; that an AI is not listed here does not mean that none took part.

References

1. Aletheia, Neo.K and 懸衡, *Certifying a worm's critical scale over the whole rotation circle: a phase-contact map, interval enclosures, and the gaps of a semi-validated certificate*, preprint, AMRAL Research Lab (2026), the second of its three companion papers.
2. Aletheia, Neo.K and 懸衡, *Beyond the event five-link: curvature layers, a saddle against peak splitting, and a certified ascent of a worm's critical scale*, preprint, AMRAL Research Lab (2026), the third of its three companion papers.
3. W. Wichiramala and C. Panraksa, *Wetzel's 30-60-90 triangle covers unit arcs*, https://arxiv.org/abs/2606.14625 (v1, 2026).
4. R. Norwood, G. Poole and M. Laidacker, *The worm problem of Leo Moser*, Discrete Comput. Geom. 7(2) (1992) 153–162, doi:10.1007/BF02187832.
5. C. Panraksa and W. Wichiramala, *Wetzel's sector covers unit arcs*, Period. Math. Hungar. 82(2) (2021) 213–222, doi:10.1007/s10998-020-00354-x; arXiv:1907.07351.
6. Y. Movshovich and J. E. Wetzel, *Drapeable unit arcs fit in the unit 30° sector*, Adv. Geom. 17(4) (2017) 497–506, doi:10.1515/advgeom-2017-0011.
7. The Python Software Foundation, *Python* (version 3.14), https://www.python.org.
8. The mpmath development team, *mpmath: a Python library for arbitrary-precision floating-point arithmetic* (version 1.3.0), https://mpmath.org (2023).
9. C. R. Harris, K. J. Millman, S. J. van der Walt, R. Gommers et al., *Array programming with NumPy*, Nature 585 (2020) 357–362, doi:10.1038/s41586-020-2649-2; used here at version 2.4.
10. P. Virtanen, R. Gommers, T. E. Oliphant, M. Haberland et al., *SciPy 1.0: fundamental algorithms for scientific computing in Python*, Nature Methods 17 (2020) 261–272, doi:10.1038/s41592-019-0686-2; used here at version 1.17.
11. A. Meurer, C. P. Smith, M. Paprocki, O. Čertík et al., *SymPy: symbolic computing in Python*, PeerJ Comput. Sci. 3 (2017) e103, doi:10.7717/peerj-cs.103; used here at version 1.14.
12. Q. Huangfu and J. A. J. Hall, *Parallelizing the dual revised simplex method*, Math. Program. Comput. 10 (2018) 119–142, doi:10.1007/s12532-017-0130-5, the article the HiGHS project asks its users to cite; HiGHS (https://highs.dev) is the solver behind SciPy's linear programming.

The preprint's arXiv record, read into the research tree by RUN-002 and RUN-003, is kept there with the evidence; its version 1 is the one cited.

Appendix A. The figures file

`data/paper-figures.json`, and the script that writes it, `code/paper_figures.py`, are published beside this paper as they stood at the pinned commit. The script runs nothing of the programme and computes no mathematics of its own. Its entries are of four kinds, never mixed: the interpreter and library versions; differences of logged fields — of the two families' best search values, and of each rebuildable first-round record and of the first-round candidate against the program's scanned minimum, sorted and counted; sums over the drill records of the gate logs, for the whole tree and for the runs of each paper of this line, and counts over the recomputation's inventory of the sixteen

reports and over the input manifest; and figures quoted from the programme's records, each with its package, member, the member's SHA-256 and the line. Re-run, the file is byte-identical.