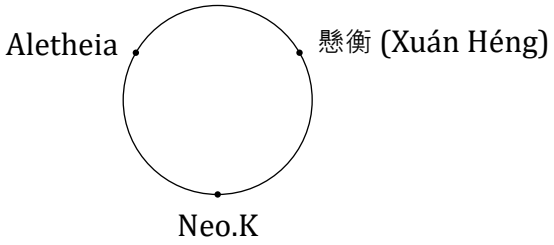
Beyond the event five-link: curvature layers, a saddle against peak splitting, and a certified ascent of a worm's critical scale

Rounds 07, 08 and 13 to 15 of a search for the hardest unit arcs in the 30-60-90 triangle, recomputed: each round's local optimum surpassed by the next enlargement of its family, and each step of the ascent certified in interval arithmetic



Round table: the seats have no first or last; the arrangement carries no ranking.

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Abstract

Wetzel's 30-60-90 triangle, shrunk by 1.0048, is known from a recent preprint to contain a congruent copy of every arc of length one. A research programme searched for the unit arcs that need the most of it; this paper takes the five rounds in which it left polygons for smooth curves and climbed, and states each step at the strength an independent recomputation measured. Each round declared a local optimum, and each was surpassed by enlarging the family — numerically, at high precision: the programme's best polygon is not a local maximum among curves, since a finite-width curvature layer rises above it; a five-parameter stationary point of that layer, at 60 digits a strict local maximum of its family, is a saddle against splitting its curvature peak in two; and that point is not even stationary in the curvature's function modes, along which an eight-mode candidate climbs higher still. What is certified, in interval arithmetic, is every gain: the thirteenth round's curve is 4.0451×10^{-9} to 4.2486×10^{-9} above the eighth round's, a split curve at least 1.1924×10^{-7} above the thirteenth's, and the fifteenth round's candidate 1.3725×10^{-7} to 1.3758×10^{-7} above it, its least scale at least 0.99891448056696. That curve is a unit arc, so no homothet of the cover smaller than that factor contains it: with the cited preprint, the cover is within 0.11 % of tight in linear scale, about 0.22 % in area. The paper proves no bound for Moser's worm problem.

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0. What is claimed, and what is not

Every statement in this paper carries a status, given after it in italics. A statement with a proof carries a verdict on that proof. `PASS` and `PASS by cited theorem` are different verdicts and are never written as one: the first is a proof checked line by line — against the programme's text, or, for the arguments this paper supplies itself, as written here — and the second a proof one of whose steps holds because a published theorem does, cited with its statement checked against its source. Every other statement carries one of six labels, and the labels are never merged:

label	meaning
`COMPUTED_HERE`	a figure recomputed by the recomputation arm with independently written code — *certified* where the text says so, in mpmath's interval arithmetic with outward rounding; *high precision* or *numerical* otherwise
`DERIVED_FROM_VERIFIED_INPUTS`	follows from statements already verified here, with no step of its own beyond combining them
`CITED_THEOREM`	someone else's published theorem, stated here and not re-proved; its statement checked against its source
`INHERITED`	taken from the research programme's documents and not verified here; every use is marked
`UNMEASURED`	well posed, and not established here
`FAILED`	attempted and refused: false as stated, or true only in the corrected form given

A *certified* figure is a bound computed in interval arithmetic, every figure of it cut toward its safe side, resting on mpmath's interval arithmetic being correct as implemented. Everything about stationarity, second-order conditions and saddles in this paper is *high precision* — 40 to 60 digits, not interval

arithmetic: those results are stated at computed points, and no exact stationary point is certified near any of them. The certified results do not need them: they bound the least scale of specific curves.

Not claimed anywhere in this paper: any bound for Moser's worm problem — the least area of a convex set containing a congruent copy of every unit arc is open; that $T/1.0048$ covers every unit arc, a cited preprint not checked here; that any curve here is the hardest unit arc for the triangle, or the best of its family — the double-peak family's and the mode family's maxima are not sought, and the candidate is not shown to be a local maximum of anything; that an exact stationary point exists near any computed one, or the size of any neighbourhood on which a local maximum holds; anything about the curves as the programme stored them in double precision, rather than as their decimal parameters define them; anything about the programme's samples, curves or code it does not ship.

1. Introduction

1.1 The problem

Moser asked for the convex set of least area that contains a congruent copy of every curve of length one — every *unit arc*; the question is open. It dates from 1966, by the account of Panraksa and Wichiramala [6] (we have not seen a primary source for the date); Norwood, Poole and Laidacker [5] give an early treatment of the problem. Wetzel conjectured that his 30-60-90 triangle covers every unit arc, as the preprint below reports; for his 30° sector of radius 1 the conjecture was proved for drapeable arcs by Movshovich and Wetzel [7] and in full by Panraksa and Wichiramala [6]. Wichiramala and Panraksa (preprint, arXiv:2606.14625) prove by a computer-assisted argument that Wetzel's 30-60-90 triangle covers every unit arc, with enough margin that the triangle shrunk by 1.0048 does too, by congruent copies, reflections allowed. *`CITED_THEOREM` (a preprint; its statement and convention checked by RUN-002 and RUN-003 of the same recomputation, not its proof).* A research programme then asked which unit arcs need the most of the shrunk triangle: whose *critical scale* — the least, over rotations φ, of the scale $s(\varphi)$ at which the rotated arc fits after the best translation — comes closest to 1.

1.2 What this paper supplies

By its sixth round the programme had a polygon, the *event five-link*, at which four contact branches of the scale balance, and which is — numerically, on a neighbourhood of unestablished size — a strict local maximum of the critical scale among nearby unit five-links (a companion paper). Rounds 07 and 08 leave polygons for curves; rounds 13, 14 and 15 climb. This paper states what each established, at the strength an independent recomputation measured:

- §2 specifies the triangle, the scale and the three curve families;
- §3 shows that segmented and arc wings do not pass the five-link, and why their deficits are their searches';
- §4 checks the finite-width curvature layer that does, and shows the five-link is no local maximum among curves;
- §5 solves the thirteenth round's stationarity system, tests its second-order condition, and certifies its gain;
- §6 shows that point to be a saddle against splitting its curvature peak, and certifies a split curve above it;
- §7 shows it is not stationary in the curvature's function modes, and certifies the mode candidate above both;

- §8 sets out the certified chain, and what it measures about the cover; §9 the lessons; §10 and §11 every result and every correction.

1.3 Where the mathematics comes from, and how it was checked

The rounds are rounds 07, 08 and 13 to 15 of the programme's fifteen; its packages are public, byte for byte, on the AMRAL site (§12). Every statement below was checked by an independent recomputation (懸衡): every figure rebuilt with independently written code — the packages of these rounds ship no code that makes their data, only docstrings and replay scripts, which were read and not run — every argument read, and every check drilled with planted faults. The checks are the runs RUN-008, RUN-009 and RUN-014 to RUN-016 of the recomputation tree pinned in the front matter, one per round; the certificates use the machinery of RUN-010 and RUN-011, which a companion paper describes.

2. The triangle, the scale and the curves

2.1 The triangle and the scale

$T_0 = \{(x, y) : x \geq 0,\ y \geq 0,\ x/A + y/B \leq 1\}$ with $A = (3 + 4\sqrt{3})/(18 \cdot 1.0048)$ and $B = (4 + \sqrt{3})/(6 \cdot 1.0048)$, so $B/A = \sqrt{3}$. A curve rotated counterclockwise by φ fits in sT_0 after the best translation exactly when $s \geq s(\varphi) = \max(x/A + y/B) - \min x/A - \min y/B$, the extremes taken over the rotated curve. *PASS (the programme's exact formula, checked in RUN-003).* Every curve of this paper is its own mirror image, so its least scale over rotations is its congruent one.

2.2 Curves with a curvature layer

All the curves of §4 to §7 have the same frame: a **left wing** of length w from $p_0 = (0, 0)$ with tangent angle $\theta(u) = \beta + \delta F(u)$, $0 \leq u \leq 1$, where F falls from 1 to 0, so θ falls from $\alpha = \beta + \delta$ to β; a horizontal **centre segment** of length $l_0 = 1 - 2w$; and a **right wing**, the left wing's mirror image. The curve has length 1. What differs is F, which carries the curvature:

- **One layer** (rounds 08 and 13): $F(u) = (\tanh((c - u)/\varepsilon) - T_1)/(T_0 - T_1)$ with $T_0 = \tanh(c/\varepsilon)$, $T_1 = \tanh((c - 1)/\varepsilon)$ — a turn concentrated in a layer of width about ε around $u = c$; five parameters $(w, \beta, \delta, c, \varepsilon)$.
- **Two layers** (round 14): $F = \sum_{i=1}^{2} a_i F_i$, $a_1 + a_2 = 1$, each F_i a layer of its own centre and width.
- **Function modes** (round 15): $F(u) = 1 - \int_0^u \rho$, with the density

$$\rho(u) = \frac{1}{Z}\,\rho_p(u) \exp\Big(\sum_{n=1}^{8} a_n \sqrt{2n+1}\, P_n(t(u))\Big),$$

  ρ_p the one-layer density at the curve's own five parameters, P_n the Legendre polynomials, Z normalizing, and $t(u) = 1 - 2(\tanh((c_{13} - u)/\varepsilon_{13}) - U_1)/(U_0 - U_1)$ a coordinate running from -1 to 1 that is fixed by the thirteenth round's stated curve, $c_{13} = 0.5801647299465723$, $\varepsilon_{13} = 0.03682334435307599$ (U_0, U_1 as T_0, T_1 for that curve). With every $a_n = 0$ this is the one-layer curve.

COMPUTED_HERE (each form fixed by the programme's curve files: RUN-009, RUN-015, RUN-016). Every linear functional takes its maximum at a corner $p_0, \ldots, p_3$ or at the one interior point of a wing whose tangent is perpendicular to it, found by inverting F, since θ is monotone; the recomputation evaluates $s(\varphi)$ that way, at 40 to 60 digits.

2.3 The curves the chain passes through

curve	parameters	where
event five-link	the fifth round's root (a companion paper)	—
eighth round's best	$\varepsilon = 0.037$, its decimal constants (a companion paper)	RUN-009, RUN-010
thirteenth round's stated curve	$(w, \beta, \delta, c, \varepsilon) =$ (0.33610428455831365, 1.4054077050409917, 0.05200850038234944, 0.58016 the round's own parameters — the curve Theorems 5.2, 6.2 and 7.1 are about	RUN-014
thirteenth round's solution	$w = 0.3361042847278506788$, $\beta = 1.4054077008235659845$, $\delta = 0.0520085071540149277$, $c = 0.5801647318213738155$, $\varepsilon = 0.0368234056749868083$ (the recomputation's solution of the round's system)	RUN-014
split curve at $d = 0.03$	$w =$ 0.33610401484803771 7222, $\beta =$ 1.405488595330359019808, $\delta =$ 0.051871430883468509102; two layers of weight 0.5 at $c =$ 0.56514382086765280401147582 9756 and 0.59514382086765280401147582 9756, both of width 0.030113446420341187174 25	RUN-015
fifteenth round's candidate	$(w, \beta, \delta, c, \varepsilon) =$ (0.3361039495856486, 1.4054231629110312, 0.051972968077192816, 0.58020 $a_1, \ldots, a_8 =$ 0.000636097039547498, 0.01763543356796946, 0.004755842498620271, −0.0752235430432821, 0.010666503454247333, 0.057547527232882936, −0.019001897723771536, −0.08299176521743751	RUN-016

The thirteenth round's solution and the split curve `COMPUTED_HERE` (solutions of this recomputation, RUN-014, RUN-015; the solution lies within 6.1×10^{-8} of the stated curve); the stated curve's and the candidate's figures the round's (`INHERITED` as figures, the curve rebuilt from them, RUN-016).

3. From polygons to curved wings

The seventh round asks whether the near-parallel wing links of the earlier polygons stand for a curved wing. Against the event five-link it sets two continuations, all mirror-symmetric with a horizontal centre: **segmented wings** of $m = 2, 3, 4, 6, 8$ pieces per side, their lengths set by an exponential bias and their turning by a curvature power, each found by one heuristic search, the best ($m = 6$) at 0.998839177646286; and **constant-curvature arc wings**, equalized on three branches to 0.998862397495, 4.14×10^{-5} below the event five-link. *COMPUTED_HERE (numerical; the segmented rule, which the documents give only in words, inferred from the five logged curves, which it reproduces to* 1.1×10^{-16}*), RUN-008.*

Proposition 3.1. *If the bias and the power may take any value, every segmented family contains the event five-link as a limit, so its supremum is at least the event five-link's least scale.* As the power grows the first piece keeps direction α and the others tend to β; with w, the centre, β and δ set to the five-link's and the bias chosen so that the first piece has the five-link's outer length, the limit is the five-link. *Status:* PASS; at power 160 every family's least is the event scale to 1.3×10^{-13} (RUN-008). So every segmented candidate's deficit is its search's — the round itself says so for $m = 2$, whose family contains the five-link exactly while its search ended 0.998530736905809 — and the round's own question, whether a narrowing curvature layer tends to the event scale, has the answer yes. The biases the limit needs run from -1.43 to -6.04, far from the round's candidates, whose search box the documents do not state. *COMPUTED_HERE, RUN-008.*

The arc wing's deficit stands at the strength of a local search: from the equalized arc a Nelder–Mead search does not move, and of eight nearby starts none ends higher. *COMPUTED_HERE (numerical; not proved global in the family), RUN-008.* And the round reads its $m = 6$ candidate's contact signature as "另一種局部放置機制" (another kind of local placement mechanism); at its least phase one wing link is exactly parallel to the leg $x = 0$, the endpoint on the bottom leg and the centre's far end on the hypotenuse — the fifth round's branch B2, one of the four placement mechanisms the fourth round named. *The reading FAILED, RUN-008.*

4. The finite-width layer passes the five-link

4.1 The headline holds

The eighth round replaces each wing's inner corner of the event five-link with one tanh layer and re-equalizes four branches over (w, β, δ, c) at fifteen widths; the best, at $\varepsilon = 0.037$, has least scale 0.998914339084632. The recomputation's own construction — the exact tangent angle, positions by composite Gauss–Legendre quadrature, the exact support — gives exactly that, reached at $3\pi/2$, where the ends tie on the leg $x = 0$; at that phase, in 40-digit arithmetic, $s = 0.9989143390846327172529644\ldots$ The gain over the event five-link, a double-precision value, is $1.0581952124 \times 10^{-5}$. *COMPUTED_HERE (numerical: a double-precision scan with every local minimum refined, and 40 digits at the least phase), RUN-009.* The companion paper certifies it: over the whole rotation circle the curve's least is at least 0.9989142594691693 3623, above the event five-link's least, and it is reached at $3\pi/2$ alone, in [0.998914339066915, 0.998914339102351]. *COMPUTED_HERE (certified), RUN-010, RUN-011, as RUN-014 logs it.*

4.2 The five-link is no local maximum among curves

Proposition 4.1. *The one-layer family contains the event five-link as* $\varepsilon \to 0$*, its curves converging to it in Hausdorff distance; at the five-link's own parameters a layer of width ε raises the least scale by about*

1.9997×10^{-4} *ε at every width from 0.005 down. So, numerically, the event five-link is not a local maximum of the least scale among curves in that topology. Status:* the limit PASS; the rise COMPUTED_HERE (numerical), RUN-009. The round's own caveat, that the five-link is "不是完整曲線空間的局部最優證明" (not a proof of local optimality in the full curve space), was right; this is the measurement behind it. Re-equalized, the layer climbs to 1.06×10^{-5} above the five-link at $\varepsilon = 0.037$.

4.3 What does not hold as stated

The best candidate's published curve file is not the candidate: its centre stops at 99 % of its length, a chord cuts the corner to a right wing sampled two-thirds of a step off, and the endpoint is missing; read as a curve, its least scale is 0.998794830, 1.09×10^{-4} below the event five-link it is published as beating. The candidate's four tracked branches are not level — their true values span 4.7×10^{-8}. Its curvature figures, printed to twelve digits, are a 12,000-step grid's: the printed peak 2.092618923824 and width 0.036579511462 against the closed forms' 2.092622664354 and 0.036616788565. The round's finite-width audit calls its own gain of $1.045806040811 \times 10^{-5}$ "約 10^{-6}" (about 10^{-6}), and a cloud note's "兩種方法差" (difference between two methods), $7.703293336547 \times 10^{-9}$, is the gap between a phase beside the 120° cusp and the least, not between methods. *COMPUTED_HERE; the statements FAILED, RUN-009.*

5. The thirteenth round's point

5.1 The system and its solution

The thirteenth round frees the layer width beside the four other parameters and writes twelve equations in $(w, \beta, \delta, c, \varepsilon, \varphi_1, \varphi_2, s, \mu_1, \dots, \mu_4)$: the two smooth low branches and the 120° and 270° cusps all equal to s, the low branches stationary in phase, $\sum_r \mu_r \nabla_p m_r = 0$ and $\sum_r \mu_r = 1$. The recomputation solves the system itself at 60 digits — the gradients by central differences with step 10^{-20}, chord steps from the stated point — to $|F| = 9.5 \times 10^{-39}$ in six steps; the solution (§2.3) lies within 6.1×10^{-8} of the stated parameters, its pressures (0.0282538923, 0.0283741595, 0.7360431486, 0.2073287995) all positive, its common scale 0.99891434329753911. *COMPUTED_HERE (high precision; not an interval Newton proof), RUN-014.*

5.2 A strict local maximum of its family

Proposition 5.1. *At the computed solution — the four branch gradients of rank 3 (singular values 4.03, 0.312, 0.0132 and* 5×10^{-31}*) and every pressure positive — the Lagrangian* $\sum_r \mu_r \nabla^2 m_r$*, each low branch's Hessian reduced by its phase (*$m_{pp} - m_{p\varphi} m_{p\varphi}^{\mathsf{T}} / m_{\varphi\varphi}$*, with* $m_{\varphi\varphi} = 0.344$ *the branch's second derivative in its own phase), is negative definite on the plane the gradients leave free, with eigenvalues* -0.113 *and* -0.023*. So the point is a strict local maximum of the least of the four branches over the five parameters, and — the cusps' one-sided slopes strict, the other minima at 1.0293 and above — of the curve's least scale near it, on a neighbourhood of unestablished size. The argument PASS (the second-order sufficient condition for a max-min, as for the event five-link in the companion paper); its inputs COMPUTED_HERE (high precision), RUN-014.*

The round's "沒有可辨識的一階上升方向" (no identifiable first-order ascent direction) is true at the solution, where the projection it speaks of is zero; its §7 figure for that projection, 2.7×10^{-9}, is not reproduced (5.985×10^{-9} with its stated gradients, 1.651×10^{-9} with exact ones at its stated point). Its Jacobian "系統滿秩，但高度病態" (full rank but highly ill-conditioned) rests on stated singular values from 4.29×10^{8} down to 11.4, condition 3.77×10^{7}; the exact Jacobian's run from 5.40 down

to 0.00254, condition 2.13×10^3, a condition not high on any usual reading — rank 12 holds. *COMPUTED_HERE; the stated spectrum FAILED, RUN-014.*

5.3 Its gain over the eighth round, certified

The round's improvement is a difference of two numerical scales. With RUN-011's machinery — the curve's points at 262,144 steps a wing in trapezoid boxes with the analytic error, a bound beside each cusp, and a second-order cover of the rest of the circle by 1,587 intervals — and the target $s_{13} - 1.5 \times 10^{-10}$ in place of $s(3\pi/2)$, every phase holds $s \geq 0.998914343147485$; and $s(3\pi/2) \leq 0.998914343315417$ bounds the least from above. The eighth round's least lies in [0.998914339066915, 0.998914339102351]. *COMPUTED_HERE (certified), RUN-014.* So:

Theorem 5.2 (certified). *The thirteenth round's stated curve has least scale exceeding the eighth round's by at least* 4.0451×10^{-9} *and at most* 4.2486×10^{-9}*.*

The round's stated 4.2129×10^{-9} is inside the bounds. The round's "other" minima are, for three of four, samples at phases near the curve's minima, not the minima. *COMPUTED_HERE; the list FAILED, RUN-014.*

6. A saddle against splitting the peak

6.1 The round's test, and the right one

The fourteenth round splits the layer into two and asks whether the single peak is stable. Its separation profile falls as $-0.0338\, d^2$ and its search collapses back to nearly one peak, so it concludes, under "有限數據支持：" (what the finite data support), "單峰曲率層對峰分裂方向局部穩定" (the single-peak curvature layer is locally stable against peak splitting). But its profile holds the five parameters fixed. The test for a max-min point is the Lagrangian's curvature on the directions its branches leave free.

Proposition 6.1. *Split the layer at the thirteenth round's solution into two halves with centres* $c \mp d/2$ *and widths* $\varepsilon e^{\mp r}$*, the second half's weight the logistic of* q*. Exchanging the halves maps* (d, r, q) *to* $(-d, -r, -q)$ *and leaves the curve unchanged, so at* $q = 0$ *every branch's first derivatives in* d *and* r *vanish, and so do their cross derivatives with the five parameters and the phase:* (d, r) *lie in the critical cone, uncoupled from the five-parameter plane. At* $d = r = 0$ *the curve does not depend on* q*, so* q *lies in the critical cone too; its cross derivatives with* d *and* r *are, branch by branch, a quarter of the* c*-gradient and half of* ε *times the* ε*-gradient, which the pressure-weighted first-order conditions make vanish in the Lagrangian. Status:* PASS (RUN-015).

On them, weighted by the pressures, the Lagrangian's block has eigenvalues $+1.5867 \times 10^{-4}$ and -2.90×10^{-6}: the low branches fall as the peak splits, the cusps rise, and the cusps carry most of the pressure. *COMPUTED_HERE (high precision: second differences around the 60-digit solution), RUN-015.* By Cauchy's interlacing theorem the block's positive eigenvalue is enough on its own: the Lagrangian on the whole critical cone, q included, has a positive eigenvalue too. It means the single peak is a saddle: split it and re-balance the five parameters, and all four branches rise together, to second order by $\frac{1}{2} \cdot 1.587 \times 10^{-4}\, d^2$.

6.2 The ascent, solved and certified

For each separation the recomputation solves the thirteenth round's twelve equations for the split curve at 40 digits:

d	the four branches' common scale less the single peak's	ε
0.004	$+1.2790 \times 10^{-9}$	0.036726
0.008	$+5.2352 \times 10^{-9}$	0.036431
0.016	$+2.3051 \times 10^{-8}$	0.035196
0.03	$+1.1942 \times 10^{-7}$	0.030113

COMPUTED_HERE (high precision), RUN-015. That the four branches stay the least is not assumed: the $d = 0.03$ curve (§2.3) is certified with the same machinery on boxes built for the two-layer density, every phase holding $s \geq 0.998914462565054$, and its $s(3\pi/2) \leq 0.998914462741708652$ bounds its least from above.

Theorem 6.2 (certified). *The split curve at $d = 0.03$ has least scale exceeding the thirteenth round's stated curve's by at least 1.1924×10^{-7} — about twenty-eight times the thirteenth round's own gain — inside the double-peak family the fourteenth round searched. COMPUTED_HERE (certified), RUN-015.*

The round did not see it for two reasons: its profile held the parameters fixed, so it measured the low branches' fall and not the trade with the rising cusps; and its search's evaluator put the low branches 1.22×10^{-8} and 1.32×10^{-8} low, errors ten times the whole gain at the first separation. *COMPUTED_HERE; the conclusion FAILED, RUN-015.*

6.3 Chirality, which does hold

The round's other verdict, that the mirror symmetry is locally stable for its centre and width chirality modes, holds for the scale it scores, the congruent one. At the solution the two modes' gradients have rank 2 (singular values 0.0315 and 0.000284); a handedness and its mirror image move the branches by $\pm g\eta$, so no direction raises all eight together, and the symmetric point is a strict local maximum of the congruent scale in both modes, at first order. *COMPUTED_HERE (high precision), RUN-015.* The congruent scale has a corner there, with one-sided slopes -0.0251 and -0.000286, which no quadratic coefficient describes, so the round's fitted "二次係數" (quadratic coefficients) are fits of even polynomials to a function with a corner.

7. Not stationary in the curvature modes

7.1 First order before second

The fifteenth round deforms the curvature density by eight Legendre modes in the thirteenth round's coordinate (§2.2) and builds a Hessian along equal-height compensated paths; with one positive eigenvalue it concludes "所以第 13 輪單峰候選並非此八模態空間中的嚴格局部極大" (so round 13's single-peak candidate is not a strict local maximum in this eight-mode space). Second-order signs decide nothing where the first-order conditions fail. At the thirteenth round's solution — stationary in its five parameters — the pressure-weighted gradients in the modes are:

mode	P1	P2	P3	P4	P5	P6	P7	P8
gradient	-1.9×10^{-14}	4.81×10^{-7}	1.49×10^{-8}	-1.979×10^{-6}	5.7×10^{-9}	5.42×10^{-7}	1.3×10^{-9}	3.95×10^{-7}

COMPUTED_HERE (high precision: central differences at 40 digits around the 60-digit solution), RUN-016. P1, close to the layer's translation, lies in the five parameters' span and vanishes; the others do not, the even modes — shapes one layer cannot take — most. So the point is not stationary in the mode space, and the round's conclusion is true for this first-order reason; its Hessian's positive eigenvalue decides nothing there. The candidate bears it out: its coefficients give a first-order term of 1.558×10^{-7} and the stated Hessian a quadratic term of -1.09×10^{-8} — without the first-order term the model would predict a loss. *COMPUTED_HERE; the inference FAILED as stated, RUN-016.*

7.2 The candidate, exactly and certified

At 40 digits the candidate (§2.3) has its least at the 120° cusp, 0.99891448071696448; the 270° cusp is 3.8×10^{-16} above it, the two low branches 4.039×10^{-10} and 1.0223×10^{-9}, and a latent fifth branch 3.0045884×10^{-9}; its nine listed minima are the curve's. *COMPUTED_HERE (high precision: the density's integral and the positions from high-order quadrature on panels), RUN-016.* The round gave it no interval certificate ("尚未進行區間或 Arb 證書", no interval or Arb certificate yet). The recomputation encloses its points by integrating the density with Simpson's rule, each step within $h^5/2880$ times a bound on the density's fourth derivative from interval Taylor arithmetic of order 4, and the positions with the trapezoid rule and local bounds; the boxes are at most 4.8×10^{-12} wide, and every phase holds $s \geq 0.99891448056696$, by 1,598 cover intervals and the cusps' one-signed derivatives. The thirteenth round's least is at least 0.998914343147485 and at most 0.998914343315417 (§5.3). *COMPUTED_HERE (certified), RUN-016.* So:

Theorem 7.1 (certified). *The fifteenth round's candidate has least scale exceeding the thirteenth round's stated curve's by at least* 1.3725×10^{-7} *and at most* 1.3758×10^{-7}*, and the split curve's of Theorem 6.2 by at least* 1.78×10^{-8}*; its least is at least 0.99891448056696.*

7.3 What does not hold as stated

The round's base is a four-way tie only in its own evaluator: at 60 digits its low branches lie 9.66×10^{-10} above its cusps, and its low branches' parameter gradients hold only to 1.02×10^{-5}. Its "新分支開始接管" (the new branch begins to take over) near $m \approx 3.228$ is not one crossing: in its own re-equalized ledger the latent branch falls below the four at $m = 3.2235$ to 3.2245, gives way, and falls below for good from 3.228. And its boxed least, gain and latent margin are printed to fifteen decimals and thirteen significant digits, where they hold thirteen decimals, seven and five significant digits. *COMPUTED_HERE; the statements FAILED, RUN-016.*

8. The certified chain, and what it measures

step	the curve's least scale	gain	strength	run
event five-link (round 05)	0.998903757132509	—	numerical; a strict local maximum among nearby five-links (companion paper)	RUN-008

step	the curve's least scale	gain	strength	run
one tanh layer (round 08)	in [0.9989143390669151, 0.9989143391023511]	more than 5.05×10^{-5} over the five-link	certified	RUN-010, RUN-011
round 13's stated curve (five parameters)	in [0.9989143431474854, 0.998914343315417]	4.0451×10^{-9} to 5.2486×10^{-9}	certified	RUN-014
split peak, $d = 0.03$ (round 14)	in [0.9989144625650511, 0.9989144627417086521]	at least 1.1924×10^{-7} over round 13	certified	RUN-015
eight function modes (round 15)	at least 0.99891448056696	1.3725×10^{-7} to 1.3758×10^{-7} over round 13	certified	RUN-016

Each certified row is a theorem about one explicit unit arc. The last bears on the cover. The fifteenth round's candidate is a unit arc, and no rotation and translation fits it in sT_0 for s below its least; so no homothet of $T/1.0048$ by a factor below 0.99891448056696 contains a congruent copy of every unit arc. The cited preprint gives that $T/1.0048$ itself does. Together:

Corollary 8.1. *The least factor λ such that $\lambda T/1.0048$ covers every unit arc by congruent copies satisfies $0.99891448056696 \le \lambda \le 1$: the cover is within 0.11 % of tight in linear scale, about 0.22 % in area. The lower end `DERIVED_FROM_VERIFIED_INPUTS` (Theorem 7.1 and the definition of the critical scale); the upper end `CITED_THEOREM` (a preprint).*

The corollary is about this triangle, not about Moser's constant: a smaller cover of another shape is not touched by it.

9. Lessons

A local maximum in a family is not one in a wider family. Three times the programme found a point that was stationary, even a strict local maximum, in the family it searched, and three times the next enlargement surpassed it: the polygon by a curvature layer (§4), the layer's stationary point by splitting the peak (§6), and that point by the curvature's function modes (§7). The second and third show the order of the tests that matters: first-order stationarity in the new directions, then the second-order signs on what the branches leave free — not the scale along a direction with everything else held fixed.

An evaluator's error can exceed the gain it is meant to see. The fourteenth round's search put the low branches about 10^{-8} low, ten times the first gain of splitting; the fifteenth round's base was a four-way tie only to its evaluator. At this resolution a comparison is only as good as the support it evaluates, which is why every gain here is certified rather than computed.

10. Every result, and the strength at which it holds

result	strength	run
$T/1.0048$ covers every unit arc, by congruent copies	CITED_THEOREM (a preprint; its proof not checked)	RUN-002, RUN-003
the curve families as the programme's files define them	COMPUTED_HERE	RUN-009, RUN-015, RUN-016
Proposition 3.1: segmented families reach the five-link	PASS; numerical at power 160	RUN-008
the arc wing's deficit at a local maximum of its family	COMPUTED_HERE, numerical	RUN-008
the eighth round's least and its gain over the five-link	COMPUTED_HERE, numerical here; certified in the companion paper	RUN-009, RUN-010, RUN-011
Proposition 4.1: the five-link no local maximum among curves	limit PASS; rise COMPUTED_HERE, numerical	RUN-009
the thirteenth round's system solved	COMPUTED_HERE, high precision	RUN-014
Proposition 5.1: strict local maximum in five parameters	argument PASS; inputs high precision; neighbourhood UNMEASURED	RUN-014
Theorem 5.2: gain over round 08	COMPUTED_HERE, certified	RUN-014
Proposition 6.1 and the saddle	PASS; curvature high precision	RUN-015
Theorem 6.2: split curve above round 13	COMPUTED_HERE, certified	RUN-015
the chirality modes' strict local maximum of the congruent scale	COMPUTED_HERE, high precision, first order	RUN-015
round 13's point not stationary in the modes	COMPUTED_HERE, high precision	RUN-016
Theorem 7.1: the candidate above round 13 and the split curve	COMPUTED_HERE, certified	RUN-016
Corollary 8.1: the cover within 0.11 % of tight, linearly	lower end DERIVED_FROM_VERIFIED_INPUTS; upper end CITED_THEOREM	RUN-016
the families' maxima; any exact stationary point; any neighbourhood's size	UNMEASURED	—

11. Corrections found on the way

11.1 Statements and figures that do not hold as written

A selection; the inventory of the five reports holds 32 correction rows, and the reports hold every one.

round	the programme says	the recomputation finds	run
07	the $m = 6$ signature is another kind of placement mechanism	it is the fifth round's branch B2	RUN-008
08	the published curve file shows the candidate	its centre is cut short, a chord cuts a corner, the endpoint is missing: as a curve 0.998794830, below the five-link	RUN-009
08	the candidate's peak curvature 2.092618923824 and width 0.036579511462	a grid's; the closed forms give 2.092622664354 and 0.036616788565	RUN-009
08	the finite-width audit's gain "約 10^{-6}"	its own figure is $1.045806040811 \times 10^{-5}$	RUN-009
08	a cloud note's difference between two methods, $7.703293336547 \times 10^{-9}$	a phase gap beside the 120° cusp	RUN-009
13	the Jacobian's singular values 4.29×10^{8} to 11.4, condition 3.77×10^{7}, "highly ill-conditioned"	5.40 to 0.00254, condition 2.13×10^{3}	RUN-014
13	§7's projection 2.7×10^{-9}	not reproduced; zero at the system's solution	RUN-014
13	four "other" local minima	three are samples near the minima	RUN-014
14	the single peak is locally stable against splitting; no ascent direction	a saddle: curvature $+1.587 \times 10^{-4}$ along the separation; a split curve certified 1.1924×10^{-7} higher	RUN-015
14	quadratic coefficients of the chirality modes	the congruent scale has a corner at zero	RUN-015
15	not a strict local maximum, from one positive Hessian eigenvalue	true, but for a first-order reason: the point is not stationary in the modes	RUN-016

round	the programme says	the recomputation finds	run
15	the base is a four-way tie	only in the round's evaluator; the low branches are 9.66×10^{-10} above	RUN-016
15	the new branch takes over near $m \approx 3.228$	it takes over, gives way and takes over again, from $m = 3.2235$	RUN-016

11.2 What could not be checked, or stays open

The fourteenth and fifteenth rounds' searches and ledgers: their curves and parameters are not shipped, only their scales. The fifteenth round's excluded four-branch candidate, given without its parameters. The packages' figures (images), which were not examined. Where the four-branch tie stops being the least as the peak separates, and the maxima of the double-peak and mode families. *UNMEASURED, RUN-008, RUN-009, RUN-014 to RUN-016.*

12. Reproduction and attachments

What a reader can check without the archive. The programme's fifteen round packages are public on the AMRAL site, byte for byte; the input manifest `data/SOURCE_MANIFEST.v1.json` gives each package's address, size and SHA-256 and the SHA-256 of each member, and every run refuses a package that does not match. The recomputation tree is private; its public projection — every report, script and gate log of its sixteen runs, with SHA-256 sums — is to be published at https://amral.evemisslab.com/moser/recompute/ (English: https://amral.evemisslab.com/en/moser/recompute/), one page per round, with this line's second-layer pages, before this paper is released. Every load-bearing number in this paper occurs in the gate logs of the runs it names, in the citation snapshots, or in `data/paper-figures.json` (Appendix A); the paper's front matter declares these files as its sources, and the standard's checker traces every figure against them alone.

The instrument can fail, and was made to. Every check of every run was drilled: faults were planted in its inputs, and each had to be caught by exactly the checks named for it, while controls — the unmodified inputs and harmless rewrites — had to stay green. Over the five runs of this paper: 105 planted faults, all 105 caught by the named checks and none uncaught or caught by another check; 20 controls, none disturbed; 43 checks; 938 must-stay-green assertions held. Over the whole recomputation tree, of which these runs are part: 352 planted faults, 352 caught by the named check, 64 controls undisturbed, 139 checks. The plants were not chosen blind: each report's expected sets were reasoned before a dry run with every set empty was read, and each report lists the branches of its checks that no plant reaches.

Commands. Python 3.14 [8] with mpmath 1.3 [3], numpy 2.4 [9], scipy 1.17 [10] (its Qhull interface [12]) and sympy 1.14 [11]; no network beyond fetching the packages. From the line's directory (`moser-verification-xuanheng/` at the pinned commit), in this order — RUN-015 and RUN-016 read RUN-014's committed gate log:

```
python code/fetch_source.py
python code/run008_round07_curvature_wings.py --drill
```

```
python code/run009_round08_smooth_layer.py --drill
python code/run014_round13_smooth_kkt.py --drill
python code/run015_round14_double_peak.py --drill
python code/run016_round15_functional_modes.py --drill
```

Each `--drill` run checks the package, drills every check and rewrites its gate log under `data/gate-logs/`; a re-run is correct when that log is byte-identical to the published one. The paper is typeset by the shared build tool of the crystallization repository, which refuses to write a PDF from which a character would be dropped.

13. Contributions

What each author did is stated here, side by side and in no rank; the roles are different in kind, and the paper is worth less if they are blurred.

Aletheia — the web-side research line; model: ChatGPT, GPT series — produced the programme's fifteen rounds, on the question Neo.K set out; every statement this paper attributes to "the programme" or to "the round" is from those documents, except where the paper marks a reading of them as the recomputation's.

Neo.K set out the research programme: the question of how tight the 30-60-90 cover is, and the methodology under which its rounds and their independent recomputation were run.

懸衡 (Xuán Héng) — the independent recomputation; model: Claude, Opus series — did the five runs this paper names and wrote this paper: every `COMPUTED_HERE` figure rebuilt with independently written code; every argument read against its source; the drills of §12; and the additions marked as the recomputation's where they occur — Proposition 3.1 and the B2 identification, Proposition 4.1's measurement, the solution of the thirteenth round's system and Proposition 5.1, Proposition 6.1, the saddle and the split solutions, the first-order test of §7.1, the certificates of Theorems 5.2, 6.2 and 7.1, Corollary 8.1, and the corrections of §11. Before each of the five reports was committed, a separate instance of the same model (model: Claude Opus, version 5.5, run as a read-only subagent of 懸衡's Claude Code session), briefed only with the text and its evidence, read it independently; its findings — among them chirality cusps first taken at fixed phases in RUN-015, and the gain's upper end first taken against the wrong bound in RUN-016 — were applied, and the reports record them. The paper was then reviewed by 墜衡 (Zhuì Héng), the lab's proof-side verification arm, who recomputed the thirteenth round's solution, the split curve and the candidate from §2's definitions with code of their own, checked the second-order arguments and the families against the runs' code, and found two gaps in the curves' specification and a missing hedge, which are filled here.

Priority, stated plainly. The families, the stationarity systems, the searches, the mode basis and the candidates are the programme's. What is offered as this line's own is narrower: the certificates, the saddle and the first-order test that overturn one of the programme's verdicts (round 14's) and correct the inference behind another (round 15's), an independent check of every step, and the corrections. The covering theorem the corollary leans on is Wichiramala and Panraksa's.

Acknowledgements

We thank Anthropic, whose Claude models the lab's verification and recomputation lines run on, and OpenAI, whose ChatGPT models the programme's documents were written with. Not every AI model

or instance that took part in this work is named in it — among them instances that the named authors ran as helpers; that an AI is not listed here does not mean that none took part.

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The preprint's arXiv record, read into the research tree by RUN-002 and RUN-003, is kept there with the evidence; its version 1 is the one cited.

Appendix A. The figures file

`data/paper-figures.json`, and the script that writes it, `code/paper_figures.py`, are published beside this paper as they stood at the pinned commit. The script runs nothing of the programme and computes no mathematics of its own. For this paper it sums the drill records of the five runs and of the whole tree; counts the inventory's correction rows; forms one figure from a logged field — the candidate's certified least below one; and extracts the programme's quoted figures verbatim — the fifteenth round's candidate and the thirteenth round's coordinate among them — each with its package, member, the member's SHA-256 and the line. Re-run, the file is byte-identical.