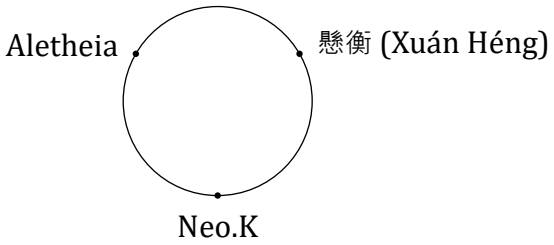
Does the Reuleaux heptagon raise the three-body value? A far-set route and how far its certificates go

An exact gate, erosion certificates in exact arithmetic, a gate that is not established, and a configuration below 0.835, on Lebesgue's universal covering problem



Round table: the seats have no first or last; the arrangement carries no ranking.

2026-10-18

Abstract

The published lower bound 0.8344 for Lebesgue's universal covering constant is a certified computation on three test bodies, the disc and the Reuleaux triangle and pentagon. Does a fourth, the Reuleaux heptagon, raise their value? By the exchange theorem it does if the heptagon misses every minimizing cover of the three. A research programme set out to prove this by two gates: near the located optimum, the heptagon misses the optimal hull by a margin; far from it, every configuration of the three bodies has hull area above the three-body ceiling. We report how far the programme's certificates go, each replayed by an independent recomputation. The near gate holds, by a finite dual certificate re-enclosed in 400-bit integer intervals, though its shipped verifier skips two premises the bound rests on. The far gate is not certified: in the branch the programme attacks, 644 of 22,516 frontier roots are closed by erosion certificates whose 56,149 area leaves all hold in exact arithmetic, and at the hardest root six more certificates hold, 182,438 area leaves in all; but a symmetrization there drops the branch's own constraint, leaving one side of a slice uncovered, and 14.6 % of that root's box, inside the branch, was closed only by a search. The least configuration found in that root, located to 80 digits, has hull area 0.834986110559224, above the ceiling by 2.05e-4, so the far gate holds there numerically. The far set does hold configurations with hull area below 0.835, numerically, the programme's headline target; so this route, which uses the heptagon only near the optimum, cannot certify 0.835 without lifting a witness far from it as well. We also give what holds of the programme's local analysis of the hardest root's configuration: its conclusions numerically, several of its arguments not as given. No bound on the constant is claimed.

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0. What is claimed, and what is not

Every statement in this paper carries a status, given after it in italics. A statement with a proof carries a verdict on that proof. `PASS` and `PASS by cited theorem` are different verdicts and are never written as one: the first is a proof checked line by line, the second a proof one of whose steps holds because a published or classical theorem does, cited with its hypotheses checked. A verdict is on the proof's own steps; a numbered result it uses keeps its own verdict. Every other statement carries one of six labels, and the labels are never merged:

label	meaning
`COMPUTED_HERE`	recomputed by the recomputation arm with independently written code — *exact* where the text says so (integer or rational arithmetic, or directed integer intervals), *numerical* where it says so (floating point, including high-precision floating point)
`DERIVED_FROM_VERIFIED_INPUTS`	follows from statements already verified here, with no step of its own beyond combining them
`CITED_THEOREM`	someone else's published theorem, stated here and not re-proved
`INHERITED`	taken from the research programme's records and not verified here; every use is marked
`UNMEASURED`	well posed, and not established here
`FAILED`	attempted and refused: false as stated, or true only in the corrected form given

Where a run grades a statement of the programme `PARTIAL` — true of part of what it says — this paper says `PARTIAL` too, and names the part that fails. A *search value* is the area at a configuration a search found: an upper bound on a minimum, never the minimum. The *hull area* of a configuration is computed by the recomputation's analytic hull — the convex hull of the disc's and the Reuleaux

bodies' arcs and corners, located in closed form and evaluated in floating point — and is numerical; only the certificates' areas are exact.

Not claimed anywhere in this paper: any value or bound for Lebesgue's universal covering constant — the problem is open; that adding the Reuleaux heptagon raises the three-body value, which is the route's aim and is not established; any lower bound above the published 0.8344; and that the configuration of §6 is the least of its root, or of its branch — its area is a floating-point value, not an enclosure, and the branch has searched configurations lower still.

1. Introduction

1.1 Three bodies, and a fourth

A *universal cover* contains a congruent copy of every planar set of diameter one, and Lebesgue's problem asks for the least area a_{Leb} of a convex one. Every lower bound so far fixes a finite family of test bodies and bounds from below the least area $\Lambda(\mathcal{F})$ of a convex set holding a congruent copy of each. The published lower bound, 0.8344 (Mishra [5]), is a certified computation on the disc D of diameter one, the Reuleaux triangle B_3 and the Reuleaux pentagon B_5; the published upper bound is 0.8440935944 (Gibbs [4]); both are cited, their arXiv records checked. For the three-body family the companion paper [1] gives

$$0.8344 \le \Lambda(D, B_3, B_5) \le 0.83478095,$$

the lower end Mishra's, the upper end a placement found by search. Mishra locates the optimum at a configuration whose hull U_0 has area 0.834780945912. *Status:* `CITED_THEOREM` for the bounds; the rest `COMPUTED_HERE`, numerical, the upper end a search value (RUN-002, RUN-009, RUN-044).

The natural next body is the Reuleaux heptagon B_7. Does it raise the value, $\Lambda(D, B_3, B_5, B_7) > \Lambda(D, B_3, B_5)$? By the exchange theorem ([1], Theorem 7.3) it does if B_7 fails to fit every minimizing cover of the three bodies — and only then, trivially. The research programme examined here, whose headline target is $167/200 = 0.835$ (the line's first paper [3]), set out to prove this smaller fact first, by a route that works near the located optimum and far from it separately, and to certify each step. This paper reports how far the certificates go.

1.2 The route

Write a three-body configuration as $v = (t_3, \rho_5, t_5)$ — the triangle's centre, the pentagon's turn and its centre, with the disc fixed at the origin and the triangle's angle fixed by the rotation of the whole — and $A(v)$ for the area of its hull $U(v)$; R_n is the circumradius of B_n. For a body K and a convex set U, $M_U(K)$ is the signed margin of [1], Definition 3.1: positive when no congruent copy of K fits in U, and then the least amount by which the best placement sticks out. Measure the distance of v from the orbit of the optimum's configuration v_0 under the problem's symmetries by the *component-motion metric*

$$q_{\mathcal{O}}(v) = \min_{w \in \mathrm{Orb}(v_0)} \max\left\{ \|t_3 - t_3^w\|,\ \|t_5 - t_5^w\| + 2R_5 \sin\tfrac{d(\rho_5, \rho_5^w)}{2} \right\},$$

where d is the distance between turns on the circle; it bounds the Hausdorff distance of $U(v)$ from the orbit of U_0 ([1], Lemma 8.4 and Corollary 9.2). Let $A_3^+ = 0.834781191$, an upper bound for $\Lambda(D, B_3, B_5)$ (§1.3). The route has two gates, with a constant $s_A > 0$:

- **Gate A** (near): $M_{U_0}(B_7) > s_A$ — the heptagon misses the optimal hull by a margin. The margin is 1-Lipschitz in the cover in the Hausdorff metric ([1], Proposition 5.7), so B_7 then misses every hull $U(v)$ with $q_O(v) < s_A$.
- **Gate B** (far): every configuration with $q_O(v) \geq s_A$ has $A(v) > A_3^+$. Equivalently, the *far-set minimum* $A_{\text{far}}(s_A) = \min\{A(v) : q_O(v) \geq s_A\}$ exceeds A_3^+.

Proposition 1.1 (the route). *If both gates hold,* $\Lambda(D, B_3, B_5, B_7) > \Lambda(D, B_3, B_5)$. A minimizing three-body cover is the hull of a configuration ([1], §7.1) and has area at most A_3^+; by Gate B that configuration lies within s_A of the orbit; by Gate A the heptagon misses its hull; so the heptagon misses every minimizing cover, and by the exchange theorem the value rises. □ *Status:* `PASS` (RUN-044, RUN-045).

The increase this would give is strict, not quantified: the route proves that the heptagon matters, not by how much. It is not a route to 0.835, and §6 shows why it cannot be made one without more.

1.3 The ceiling, and what its last digits measure

$A_3^+ = 0.834781191$ is the area of an outer polygon with 10^6 uniform supporting lines around U_0, so it bounds $\Lambda(D, B_3, B_5)$ from above. It exceeds U_0's hull area by 2.45×10^{-7} — far more than the 10^{-11} a smooth body would lose at that density — and that hull is itself a cover, so the rounding of A_3^+'s last printed digit cannot matter: $\Lambda \leq A_3^+$, all that Proposition 1.1 needs, holds with room to spare. The excess is first order: it sits at the hull's straight bridging edges, whose normals the uniform grid misses (the companion paper [2], Proposition 5.1). With the normals at the hull's 17 hand-overs — the directions where the supporting body changes — added to 10^4 uniform ones, the excess falls from 3.4×10^{-6} to 1.2×10^{-8}. So the programme's statement that the ceiling and the optimum "differ by only 2.45×10^{-7}", and that the sublevel $A \leq A_3^+$ is therefore extremely thin, measures the ceiling's construction, not the optimum (§3.3). *Status:* `COMPUTED_HERE`, numerical; the programme's statement `PARTIAL` (RUN-044).

1.4 What this paper finds

- §2: Gate A holds, by a finite dual certificate re-enclosed in 400-bit integer intervals; its shipped verifier skips two premises the bound rests on, which hold.
- §3: the far set is set up correctly — an exact dihedral quotient, a symmetric anchor, two far branches — and both branches' searched minima lie above A_3^+.
- §4: in the branch the programme attacks, 644 of the 22,516 roots of its frontier are closed by erosion certificates whose 56,149 area leaves all hold in exact arithmetic.
- §5: at the hardest root, six more certificates hold in exact arithmetic, 182,438 area leaves in all; but one gate is not established — a symmetrization drops the branch's own constraint — and 14.6 % of the root's box, lying in the branch, was closed only by a search.
- §6: the least configuration found in that root, located to 80 digits, has hull area 0.834986110559224, above A_3^+ by 2.05×10^{-4}. The far set holds configurations with hull area below 0.835, numerically; so this route cannot be turned into one for the programme's headline target without lifting a witness far from the optimum as well.
- §7: the local analysis of that configuration — what holds of its stationarity, and what does not.

Gate B is not certified. Neither, therefore, is the increase.

1.5 Where this comes from, and how it was checked

The route, its gates and its certificates are the programme's: seventeen cycles of its closure method, UESFCM (unbounded-expansion self-referential finite closure, a methodology of Neo.K's), and the local-shape papers that followed, LESR, all produced by Aletheia. The recomputation (懸衡) checked them in the runs RUN-043 to RUN-049 and RUN-056 to RUN-061 of the tree pinned in the front matter: every certificate decoded and replayed, every area leaf certified again in exact arithmetic with the recomputation's own cores, the searched minima the cycles store searched again on the recomputation's own analytic hull, and each check drilled with planted faults. Some searches the programme reports store no configuration and could not be repeated (RUN-045, RUN-049). The seventeen cycles chain cleanly and never mark evidence as proof; the target object they all name is not in the archive, and the method's own texts state its stop rule in four different ways (RUN-043). The programme as a whole is reported in the line's first paper [3]. Each result below names the run that checked it.

2. Gate A: the heptagon misses the optimum

2.1 How much it misses by

The placement certificate of [1] (§3) bounds $M_{U_0}(B_7)$ by a linear programme over finitely many support directions and orientations. With 2,048 directions and 256 orientations over one period of B_7, the grid minimum is 0.009382983648263432, at orientation 193 — the programme's number to all seventeen digits — and less the orientation loss $\pi/(7 \cdot 256) = 0.0017531209004407328$ it gives the structural bound 0.007629862747822699: floating-point linear programmes, not an enclosure. The loss uses $R_K \leq 1$; the heptagon's own circumradius would nearly halve it, to 0.0008991. *Status:* `COMPUTED_HERE`, numerical (RUN-044).

The continuous margin, minimized over every basin of orientation with the inner problem solved until the linear programme's value and the true maximum at its translation agree to 10^{-12}, is $s_7 = 0.009384687909$, at $\varphi = 0.676813$. The programme's value, 0.00938484, lies 1.52×10^{-7} above it and at neither basin's minimum — a candidate for a lower bound should not sit above the minimum. It does no harm here, because Gate A is certified by its own means. *Status:* `COMPUTED_HERE`, numerical; the programme's figure `PARTIAL` (RUN-044).

2.2 A finite dual certificate

The programme certifies Gate A by duality. It replaces B_7 by the heptagon of its corners, which lies inside it, and bounds the support of each Reuleaux body of the cover from above by its defining unit discs, so that the target's supports are smaller and the cover's no smaller; at each of 1,024 orientations it records three directions whose weights, proportional to the sines of the gaps between them, put zero in their convex hull; and an orientation-Lipschitz bridge carries the bound between grid orientations.

Proposition 2.1 (the dual bound). *If at an orientation three directions have all gaps below π, their sine-gap weights are positive and form a convex combination with zero translation term; the weighted slack is then a lower bound for the margin at that orientation. With all orientations of the grid present, the least such bound less the Lipschitz bridge bounds $M_{U_0}(B_7)$ from below.* This is LP duality ([1], Theorem 3.2) with a feasible dual, and the bridge of [1], Theorem 3.5. □ *Status:* `PASS by cited theorem` (RUN-045).

Re-enclosed here in 400-bit integer intervals — from the placement's decimals, Machin's π and enclosed Taylor sums, every width about 7×10^{-103} — every one of the 1,024 orientations is present,

every triple's gaps are below π, and the least lower bound is 0.00927495913424859, at orientation 767. Less the bridge, $\pi/(7 \cdot 1024) = 0.000438280225110183$, that gives 0.0088366789091384 — the programme's recorded value, to its last digit. The bound is conservative: at 33 sampled orientations the true margin of B_7 itself is above it by 2.2×10^{-7} to 1.8×10^{-4}, and at orientation 767 it is 0.009458708251. The programme takes $s_A = 0.0088366$. *Status:* COMPUTED_HERE, exact (RUN-045).

2.3 What the shipped verifier checks

The certificate's recorded intervals are points — lower and upper ends equal, 20-digit decimals — and on 76 of the witnesses the recorded lower end lies above the true enclosure, by at most 3×10^{-19}. That is harmless, since the verifier trusts none of them and recomputes every value. But it never checks the two premises of Proposition 2.1: that each triple's gaps are below π — past π a weight turns negative and the dual bound fails — and that all 1,024 orientations are present, which the bridge assumes. Both hold, as checked above; the verifier would have passed a certificate in which they did not. *Status:* COMPUTED_HERE; that the verifier skips them is a finding about the verifier, not about the bound (RUN-045).

3. Gate B, set up

3.1 The quotient and the domain

The problem's dihedral symmetry acts exactly: the six images of 40 random configurations have the same hull area to 10^{-15}, and the pentagon's turn, which moves by $2\pi/3 \equiv 4\pi/15$ modulo $2\pi/5$, reduces with the reflection to the fundamental interval $[0, \pi/15]$. *Status:* PASS; the check COMPUTED_HERE (RUN-045).

The translations are bounded a priori by the farthest corner: a body placed beyond T_n spans more than A_3^+ with the disc. At A_3^+ this gives $d^\star = 0.694929333199893$, $T_3 = 0.193949748872303$ and $T_5 = 0.197117585445617$, to the last unit. The programme presents this as a new, sharp root domain. It is Mishra's Remark 20 [5], to the digit at his own threshold, and his Remark 34 reports that the smaller domain does not reduce the cost: at 0.833, 27,350,260 boxes over the smaller domain against 25,425,568 over the larger. *Status:* the bound PASS; it is Mishra's, not new (RUN-044).

3.2 The anchor and the two far branches

The located optimum is only nearly mirror-symmetric: its pentagon's residual turn is 2.4129×10^{-5}. The programme replaces it by a symmetric *anchor* U_{sym}, of area 0.834780945826, whose triangle sits on the axis at $x_3 = x_3^0$; its configuration is within 1.66×10^{-5} of the optimum's in the metric q, a bound that holds (their Hausdorff distance is 7.1×10^{-6}). So every configuration with $q_{\mathcal{O}} \geq s_A$ lies at least $s = 0.0088366 - 0.0000166 = 0.00882$ from the anchor's orbit, and closing the set of such configurations about the anchor closes Gate B. That set splits exactly into two *branches*: F_3, where the triangle lies at least s from its anchor position, $d_3 = \|(x_3 - x_3^0, y_3)\| \geq s$, and F_5, where the pentagon is at least s from its own; the disc of radius s about x_3^0 is the *excluded ball* of F_3. The nearest other copy of the anchor in its orbit is $2R_5 \sin(\pi/30) = 0.109907730484$ away. *Status:* the construction PASS; the numbers COMPUTED_HERE (RUN-045).

Searched on their boundaries, F_3's minimum is at most 0.834968955147 and F_5's at most 0.834804244389 — each 10^{-9} to 10^{-8} below the programme's values — so F_5's margin over A_3^+ is at most 2.30534×10^{-5}, the thinner of the two. On the programme's own far boundary —

configurations exactly s_A from the optimum's orbit — the search reaches 0.83496965443 in the triangle's branch. Every value stays above A_3^+. These are search values, so upper bounds on the minima; no search can establish a minimum itself. The cycles attack F_3, whose frontier §4 takes up; nothing in them addresses F_5 by certificate. *Status:* COMPUTED_HERE, numerical (RUN-045).

3.3 How sharp the optimum is

Every sampled direction out of the optimum's configuration — the coordinate directions and random ones — raises the area. Minimized over all directions, though, the least one-sided slope is 1.12×10^{-5}, along a direction made 98.7 % of the pentagon's turn, and along it the sublevel $A \le A_3^+$ reaches 9.9×10^{-4} and 7.8×10^{-4} — against at most 2.9×10^{-5} along the axes. The sublevel is not extremely thin, then; but this does not threaten Gate B: 10^{-3} of the pentagon's turn moves it by 5×10^{-4}, far inside s_A. *Status:* COMPUTED_HERE, numerical (RUN-044).

The programme also considered replacing B_5 by B_7. The family $\{D, B_3, B_7\}$ has, by search, a hull of area at most 0.8305410847, so its value cannot pass 0.8344, and the programme's rejection of it stands with more room than it gave. *Status:* COMPUTED_HERE, numerical (RUN-044).

4. The branch F_3

4.1 A frontier of 22,516 roots

The programme ran an erosion branch-and-bound over F_3 — each box closed when an inner polygon of its bodies' common cores ([2], Proposition 2.1) has area above A_3^+, or when it lies in the excluded ball — and saved its unresolved frontier: 22,516 boxes, its *roots*. The prototype that produced it is not shipped, so what it closed is not checkable, and which boxes are roots is taken from the file. On the file, most of the roots lie wholly in F_3, and the least area at a root's centre is 0.835188375167486, well above A_3^+. The programme infers that the roots persist because the bound is loose, not because the area is low; centre areas cannot show that — the hardest root holds configurations below its centre's area (§6). *Status:* COMPUTED_HERE, numerical; the inference not established (RUN-046).

A first-order bound $A(v) \ge A(c) - P(c)\,\tau(B)$ — the centre's area less its perimeter times the box's Hausdorff radius — closes none of them: the perimeter is 3.33 to 3.40 and the radius up to 0.031, and the median penalty is 3.2 times the centre's slack over A_3^+. The hull's structure is simpler than its size suggests: walking the hull's boundary at every centre gives 1,332 piece signatures but only 24 body skeletons — the cyclic order in which the bodies appear on the boundary — which fall into 12 classes under reflection, one of them holding 15,210 roots. Among the roots of that class, the 649 within 10^{-3} of A_3^+ have Taylor margins positive on 646 at first order, and the closest 27 all at second order — finite-difference diagnostics, as the programme says, not bounds. *Status:* COMPUTED_HERE, numerical (RUN-046).

4.2 Erosion certificates

Two certificates close 644 of those roots: 27 roots by 27 trees with 2,699 nodes and 1,363 area leaves in all, and 617 roots by 617 trees with 109,037 nodes, 54,786 area leaves and 41 leaves in the excluded ball. Each *area leaf* carries a lower bound from a finite set of witness points — the disc's inscribed 256-gon and, on each eroded core, its corners and points along its arcs — and each excluded leaf is shown to lie in the ball. *Status:* the decoding and every stored count COMPUTED_HERE (RUN-047).

Every area leaf, again, in exact arithmetic. Each of the 56,149 area leaves was certified here with the recomputation's exact eroded cores: contact points at 1,024 directions, each proved inside its body in 400-bit integer intervals, and the exact hull of those that pass. Every leaf certifies at that density, no candidate fails, and the least margin over A_3^+ is 2.352×10^{-5} for the first certificate and 2.277×10^{-5} for the second. The certificates' own least slacks are far thinner — 3.8×10^{-8} and 2.5×10^{-9} — because their witness sets lose about 2.3×10^{-5} against the exact cores: the trees are much deeper than an exact-core bound needs. *Status:* `COMPUTED_HERE`, exact (RUN-047).

Three things the documents do not say.

- *The split rule.* The documents say only "deterministic split rule". Two natural guesses decode trees that parse but leave boxes failing even the exact-core bound. The rule is in the verifier: split the widest coordinate by weights that change with membership — $(1.5, 1.5, R_5, 1, 1)$ while the box may still meet the excluded ball, $(0.7, 0.7, 1.5\,R_5, 1.5, 1.5)$ once it lies wholly outside. The node counts cannot tell rules apart; that every leaf then passes, and that 617 per-tree minima agree to 7×10^{-16}, does.
- *The tiling.* A child's centre $c \pm h/2$ is rounded, so two children need not cover their parent exactly; the widest gap is 4.7×10^{-18}. A configuration in a gap lies within that distance of a certified box, and moving the bodies by d changes the hull area by at most $Pd + \pi d^2$ with $P \leq 8.026$ (Mishra's perimeter bound [5]), about 3.7×10^{-17} — far below every leaf's slack. So the gaps are covered, by an argument neither the documents nor the verifiers state.
- *The package.* The second certificate's verifier loads the first's from an absolute path on the machine that wrote it, and its own package does not ship that file.

Status: `COMPUTED_HERE`; the continuity step `PASS by cited theorem` (Steiner's formula and Mishra's bound; RUN-047).

4.3 What is left

Of the 649 near roots, five resist: 3766, 6217, 8001, 8015 and 8016. A targeted search splits them further and leaves 225 boxes — 200 under root 6217 and 25 under root 3766, the programme's *subroots* — whose depths and boxes reproduce here; the parts of these roots outside the 225, and roots 8001, 8015 and 8016, are closed only in search logs whose trees are not shipped. The frontier's other roots are not addressed by any shipped certificate. *Status:* `COMPUTED_HERE` (RUN-046, RUN-047).

5. The hardest root

5.1 Six certificates

Root 6217 is a box of F_3 with the pentagon's turn $\rho_5 \in [0, \pi/120]$; its 200 subroots are those of §4.3, and the programme's later cycles call the ones still open *cells*. It closes the root in pieces:

certificate	trees	nodes	area leaves	excluded leaves	least margin, exact
124 subroots	124	32,568	16,151	195	2.597×10^{-5}
the slice $\rho_5 = 0$, as a rectangle	1	299	150	—	3.456×10^{-5}

certificate	trees	nodes	area leaves	excluded leaves	least margin, exact
eight subroots of the rotation tail	8	102,062	50,626	409	2.610×10^{-5}
five cells, against 0.8349865	5	61,487	30,467	279	2.656×10^{-5}
cell 131, a prefix tree	1	—	38,893	1,966	3.022×10^{-5}
cell 129, a prefix tree	1	—	46,151	2,439	3.183×10^{-5}

Every tree decodes with its own split rule and uses exactly its bits; every stored count is the recomputation's; every leaf's witness bound passes; and every area leaf — 66,927 in the first three, 115,511 in the last three — holds again in exact arithmetic with the recomputation's exact cores, against A_3^+ or, for the five cells, against the tighter reference 0.8349865 of §6.2. The prefix trees are complete and prefix-free. The open cells of the root go from 76 to 68 to 61. *Status:* `COMPUTED_HERE`, exact (RUN-048, RUN-049, RUN-056).

The lemmas these certificates lean on hold. When several bodies move linearly with one parameter, the area of their hull is convex in it (a linear parameter system: Fáry–Rédei [6], Rogers–Shephard [7]); the secant lemmas follow. The *two-colour* criterion — a tree whose leaves are each either excluded or above the target closes its box — is right, using one premise silently: that every configuration of the branch in the box has, along the ray from the anchor, its point on the excluded ball's circle inside the box; that holds for this root. The prefix-tree criterion is right. Along random lines of three kinds — vertical, horizontal and mixed — the second differences of the hull area are all positive, the least 2.8. *Status:* `PASS by cited theorem`; the checks along lines `COMPUTED_HERE`, numerical (RUN-048, RUN-049).

5.2 A gate that is not established

The slice $\rho_5 = 0$ is closed by a reduction: reflecting in the axis and moving the triangle and the pentagon vertically together — a linear parameter system — the hull area is convex along the motion and even, so its least value on the slice is at $y_3 = y_5 = 0$, a two-parameter rectangle, which a certificate closes. The programme records this as "Gate 6217-A: closed".

But F_3 asks that $d_3 = \|(x_3 - x_3^0, y_3)\| \geq s$. The vertical motion ends at $y_3 = 0$, and where $|x_3 - x_3^0| < s$ its end point is no longer in F_3: the reduction bounds a far configuration by a near one, and a near one can lie below A_3^+ — the optimum does. The programme keeps only $x_3 \leq x_3^0 - s$, saying the root does not meet the other side; but the root's box runs to $x_3 = -0.012122$, 0.00886 past $x_3^0 - s$. So the slice also holds configurations of F_3 with $x_3^0 - s < x_3$ and $|y_3| \geq \sqrt{s^2 - (x_3 - x_3^0)^2}$ — the *arc side* — and no step of the reduction, nor the rectangle's verifier, covers them. Of a random sample of the slice's configurations in F_3, those on the arc side fall 145 in certified subroots, 6,812 in the 68 still open, and 3,899 in none of the 200 — closed only in a search. *Status:* the claim that the gate is closed `FAILED` —

the run's verdict is "not established"; the gate itself holds numerically (below; RUN-048).

The reduction can be completed. For fixed (x_3, x_5) on the arc side the area is even and jointly convex in (y_3, y_5), so on $|y_3| \geq r$ its least value lies on $|y_3| = r$, the arc $d_3 = s$; the moved point stays in the box, which holds $y_3 = 0$ and $y_5 = 0$. That leaves three parameters, the angle α along the arc and the pentagon's translation. Minimized over the translation at 25 angles, the area is least at $\alpha = \pi$, the contact configuration of §6; elsewhere on the arc it lies above that, rising and then easing to 0.835483 at the box's edge. So the gate's statement holds numerically, with the margin of §6; what is missing is a certificate for the arc side. *Status:* the completion `PASS by cited theorem` (Rogers–Shephard); the arc side `COMPUTED_HERE`, numerical (RUN-048).

5.3 "All other regions"

The programme's later text restricts what remains to the 61 open cells, "all other regions" having been "eliminated by finite certificates". The 200 subroots fill 55.9 % of the root's box. Of the rest, 29.5 % of the box lies outside F_3 and needs no closure; but 14.6 % lies in F_3 and was closed only in the targeted search of §4.3, whose trees are not shipped — and the later ledger that prunes the 200 subroots to 195 leaves it out too. Add the arc side of §5.2. *Status:* the statement `FAILED` (RUN-049, RUN-056, RUN-058).

6. A configuration below 0.835

6.1 The triple contact

A multistart search over root 6217 ends at one symmetric configuration $v^\star$: $y_3 = \rho_5 = y_5 = 0$ and $d_3 = s$, the triangle at $x_3^\star = x_3^0 - s$. There the disc, the triangle and the pentagon support the hull at exactly $\frac{1}{2}$ in four directions $\pm\theta^\star, \pi \pm \theta^\star$, with

$$\theta^\star = \arccos \frac{1}{2(R_3 + x_3^\star)} = 0.4540452162931833,$$

and the pentagon's position $x_5^\star$ solves two contact equations, located here to 80 digits. The recomputation's own minimum over x_5 on the symmetric slice sits on $x_5^\star$ to 3×10^{-14}. The hull's area is

$$A^\star = 0.834986110559224,$$

2.049196×10^{-4} above A_3^+ — and 1.389×10^{-5} *below* 0.835. Its symmetry group is exactly a single reflection. Every constant the programme gives for this configuration — $\theta^\star$, $x_5^\star$, a third contact angle $\alpha^\star$, the three bridge lengths, the area — and the 13 intervals of its hull's signature reproduce from the contact equations alone, to every digit the programme prints. *Status:* `COMPUTED_HERE`, numerical — high-precision floating point, not an enclosure (RUN-048, RUN-049, RUN-056, RUN-057, RUN-060, RUN-061).

It is the least configuration found in root 6217, on every slice examined; it is not the least known of its branch, whose searched minimum, 0.834968955147, lies elsewhere (§3.2).

6.2 Two upper references

The programme bounds the configuration's area from above by outer polygons of support lines: 20,000 directions at 70 digits give an area below 0.835, and a padded polygon of 500,000 directions

gives 0.8349865. Both reproduce. Their excess over the hull area, 7.8×10^{-6} and 2.7×10^{-7}, is again the first-order loss at the hull's straight bridging edges: with the normals at its 17 hand-overs added, 20,000 directions come within 3.1×10^{-9} and 500,000 within 4.9×10^{-12}. So the tight reference has 3.894×10^{-7} of room, not the 6.5×10^{-8} the programme states. *Status:* COMPUTED_HERE, numerical (RUN-049).

6.3 What it means

For the route's own target nothing breaks. $v^\star$ lies 2.05×10^{-4} above A_3^+, so Gate B holds there numerically, with that margin.

$v^\star$ itself does not defeat the headline target. F_3 is drawn about the anchor with the smaller radius s, and $v^\star$ lies on its inner circle: its triangle exactly s from the anchor's, whose triangle is the optimum's to within 10^{-7}, and its pentagon nearer than that to the optimum's. So its distance from the optimum's configuration in the metric q is below s_A: it lies inside Gate A's near zone, and the heptagon misses its hull. Whether it misses by enough to raise the area to 0.835 — by at least 1.389×10^{-5} — is a quantitative question the route does not ask, and nothing here measures it. *Status:* DERIVED_FROM_VERIFIED_INPUTS from the configurations' coordinates and Gate A; the size of the lift UNMEASURED.

But the route cannot reach 0.835**.** On the far gate's own boundary — configurations exactly s_A from the optimum's orbit, where Gate A's margin is used up — the search finds hull area 0.83496965443, and F_3's boundary reaches 0.834968955147; both lie below 0.835 (§3.2). With 0.835 in place of A_3^+, Gate B fails at these configurations. So the route as built — the heptagon used only near the optimum, the three bodies' own area everywhere else — cannot certify $\Lambda(D, B_3, B_5, B_7) \geq 0.835$; that would need a witness lifted in the far set as well, and a quantified lift near the optimum. *Status:* DERIVED_FROM_VERIFIED_INPUTS, numerical: a search value below 0.835 is a configuration, so the failure is a floating-point fact; an enclosure of its area would make it exact (RUN-045).

7. Is it the least of its root? The local analysis

The programme's later papers try to show that $v^\star$ is the least configuration of root 6217 — which would close the root against A_3^+ with the margin of §6. What holds of their arguments:

On the symmetric slice. In the plane of (x_3, x_5) at $y_3 = \rho_5 = y_5 = 0$ the area is convex (Fáry–Rédei), and the contact's optimality condition there is sound with the mix of one-sided gradients the programme's Cycle 12 records, $(-0.04756, 0)$: the constraint $d_3 \geq s$ is active with a multiplier of about 0.047. A later paper argues the same stationarity from a split of the contact "pressures" into a subgradient $(g_3, 0)$ with $-g_3 = 0.1228$; that split is not a subgradient, in that same plane. Over 72 directions it misses the subgradient inequality by up to 0.0663 per unit, and at finite steps it fails outright, by 2.5×10^{-4} at step 10^{-2}. The cause is the first-order model the split rests on, which treats the hull as if each normal's support moved independently and overstates the rate in every one of the 72 directions. The conclusion survives: along every feasible direction the area rises, at least at 0.0354 per unit, and the area is convex, so $v^\star$ minimizes on $x_3 \leq x_3^\star$; across the constraint, toward the anchor, the area falls at 0.0331 per unit. *Status:* the split FAILED as a subgradient; the conclusion COMPUTED_HERE, numerical (RUN-048, RUN-057, RUN-061).

Four contacts cannot certify the shape. The programme leaves open whether a measure on the four contact directions can match the configuration's curvature, for a library of test functions containing 1 and $\cos 2\theta$. It cannot. The constant mode fixes the measure's mass to the hull's perimeter P, and the

four directions share one value of $\cos 2\theta$, so any measure on them has $\cos 2\theta$-moment $P \cos 2\theta^\star = 2.05$, where the configuration's curvature measure has 0.124. A certificate needs contacts on the arcs. *Status:* PASS (RUN-056).

The pentagon's turn. Turning the pentagon off zero, the least area rises: re-minimized at 27 turns up to $\pi/120$, the profile $m(\rho)$ is non-decreasing, with ratios $(m(\rho) - m(0))/\rho^2$ between 0.1738 and 0.2254 — the programme's stored samples are search values up to 8.3×10^{-8} high. The turn is first-order critical numerically: the one-sided rate along the re-minimized profile is proportional to the turn. The paper that first derives this does so from a finite contact model in which turning the pentagon costs +0.0293 at first order, and a later paper withdraws that derivation itself. The programme's handoff document then infers $m'(0) = 0$ from the area being even in the turn; evenness gives only $m'(0^+) = -m'(0^-)$, as the later papers also say. *Status:* the profile and the criticality COMPUTED_HERE, numerical; the derivation and the inference FAILED as given (RUN-058, RUN-059, RUN-061).

The path out of zero turn. As the pentagon turns, the triangle slides along the circle $d_3 = s$ at a rate u and the pentagon translates at (w, v), all per unit of turn; along the minimizing path these rates tend to $(u, v, w) = (5.157, -0.1892, 0)$. The programme's closed forms for them — the critical rates $u^\star = 5.15712624526841370759\ldots$ and $v^\star = -0.18922818990664097 45\ldots$, with $w^\star = 0$ — reproduce to 70 digits, and the six constants they are built from to binary64; $w^\star = 0$ is exact algebra, on the assumption, still open, that the minimizing path keeps all three of its contact structures at first order. To first order the area along the path changes by a multiple $\gamma_w w$ of the pentagon's sideways rate alone — the u and v terms cancel identically — so the path is level to first order. At the four turns examined, small pieces of the hull the programme predicted, its "secondary slivers", have width below 10^{-11}: they are ties, not pieces; and the second-order slivers along the critical rates appear at the antipodal directions, not where the programme looks for them. *Status:* the algebra PASS; the rest COMPUTED_HERE, numerical; the slivers PARTIAL (RUN-059, RUN-060).

None of this is a certificate. $v^\star$ is, numerically, the least configuration of root 6217 on every slice examined; nothing here proves it.

8. Every result, and the strength at which it holds

result	holds as	checked in
§1.1 — the three-body bracket; the optimum's hull area	the bounds CITED_THEOREM; the rest COMPUTED_HERE, numerical	RUN-002, RUN-009, RUN-044
Proposition 1.1 — the route	PASS	RUN-044, RUN-045
§1.3 — the ceiling's excess is first-order loss	COMPUTED_HERE, numerical; the thin-sublevel statement PARTIAL	RUN-044
§2.1 — the grid bound; the continuous margin	COMPUTED_HERE, numerical; the programme's s_7 PARTIAL	RUN-044
Proposition 2.1, §2.2 — Gate A's dual certificate	PASS by cited theorem; the replay COMPUTED_HERE, exact	RUN-045
§2.3 — two premises the verifier skips	COMPUTED_HERE	RUN-045

result	holds as	checked in
§3.1 — the quotient; the root domain	PASS; the domain is Mishra's	RUN-044, RUN-045
§3.2 — the anchor and the branches; their searched minima	the construction PASS; the minima COMPUTED_HERE, numerical, search values	RUN-045
§3.3 — sharpness; the replacement family	COMPUTED_HERE, numerical	RUN-044
§4.1 — the frontier; its structure; Taylor diagnostics	COMPUTED_HERE, numerical	RUN-046
§4.2 — two erosion certificates, 56,149 area leaves	COMPUTED_HERE, exact; the continuity step PASS by cited theorem	RUN-047
§4.3 — what is left of the frontier	COMPUTED_HERE	RUN-046, RUN-047
§5.1 — six certificates at root 6217, 182,438 area leaves	COMPUTED_HERE, exact	RUN-048, RUN-049, RUN-056
§5.1 — convexity of linear parameter systems; secant and closure lemmas	PASS by cited theorem	RUN-048, RUN-049
§5.2 — "Gate 6217-A: closed"	FAILED (not established); the gate holds numerically	RUN-048
§5.2 — the reduction completed; the arc side searched	PASS by cited theorem; the search COMPUTED_HERE, numerical	RUN-048
§5.3 — "all other regions eliminated by certificates"	FAILED	RUN-049, RUN-056, RUN-058
§6.1 — the triple contact; its area 0.834986110559224	COMPUTED_HERE, numerical	RUN-048, RUN-049, RUN-056, RUN-057, RUN-060, RUN-061
§6.2 — the upper references	COMPUTED_HERE, numerical	RUN-049
§6.3 — $v^\star$ inside Gate A's near zone	DERIVED_FROM_VERIFIED_INPUTS; the size of the lift UNMEASURED	RUN-044, RUN-045, RUN-048
§6.3 — the route cannot reach 0.835	DERIVED_FROM_VERIFIED_INPUTS, numerical	RUN-045
§7 — the symmetric slice: the split as a subgradient	FAILED; the conclusion COMPUTED_HERE, numerical	RUN-048, RUN-057, RUN-061
§7 — four contacts cannot certify the shape	PASS	RUN-056
§7 — the turn: profile, criticality; its derivation	COMPUTED_HERE, numerical; the derivation FAILED as given	RUN-058, RUN-059, RUN-061

§7 — the path out of zero turn	the algebra PASS; the rest numerical; the slivers PARTIAL	RUN-059, RUN-060

No result is INHERITED. No area in §6 and §7 is an enclosure.

9. Corrections found on the way

9.1 Statements that do not hold as written

- **"Gate 6217-A: closed."** The symmetrization behind it drops the branch's constraint, and the arc side of the slice is covered by no certificate as a whole — a little of it falls in certified subroots (§5.2). (RUN-048, RUN-049, RUN-056)
- **"All other regions have already been eliminated by finite certificates."** 14.6 % of root 6217's box lies in the branch and was closed only by a search, whose trees are not shipped; the ledger that prunes the root's 200 subroots to 195 leaves it out too (§5.3). (RUN-049, RUN-058, RUN-059)
- **The translation subgradient.** The pressure split $(g_3, 0)$ is not a subgradient; the constraint's multiplier is about 0.047, as the programme's own earlier cycle records, not 0.1228 (§7). (RUN-057, RUN-061)
- **"Evenness gives $m'(0) = 0$"**, in the handoff document, does not follow; criticality of the turn holds numerically. The derivation from the finite contact model, in which turning costs $+0.0293$, is withdrawn by the programme's own later paper (§7). (RUN-058, RUN-059, RUN-061)
- **Gate A's verifier** does not check two premises its bound rests on; both hold (§2.3). (RUN-045)

9.2 Figures that are search values, or other quantities

- The continuous margin s_7 is 0.009384687909; the programme's figure lies 1.52×10^{-7} above it (§2.1). (RUN-044)
- The far branches' minima are search values; both lie lower than the programme's (§3.2). (RUN-045)
- The 27 samples of the rotation profile are search values, up to 8.3×10^{-8} high; their ratios are 0.1738 to 0.2254 (§7). (RUN-048, RUN-058)
- The "secondary slivers" are ties at the minimum; those reported come from a sample 1.5×10^{-9} above it. The second-order slivers along the critical rates open at the antipodal directions (§7). (RUN-059, RUN-060)
- The ceiling's 2.45×10^{-7} over the optimum, and the tight reference's 6.5×10^{-8} of room, are outer polygons' first-order losses at bridging edges; the room is 3.894×10^{-7}, and the sublevel is about 10^{-3} long along the pentagon's turn, not extremely thin (§1.3, §3.3, §6.2). (RUN-044, RUN-049)
- The printed $u^\star$ is 5×10^{-14} off, binary64 through a system whose determinant is about 10^{-4}. (RUN-060)

9.3 Credit and form

- The "sharp root domain" is Mishra's Remark 20, and his Remark 34 reports that it saves no boxes (§3.1). (RUN-044)

- The erosion certificates' split rule is stated in neither document and is recoverable only from the verifier; their rounded splits leave gaps of up to 4.7×10^{-18}, closed by a continuity argument neither states; and the second certificate's verifier cannot run from its own package (§4.2). (RUN-047)
- The recorded witness intervals of Gate A's certificate are points, 76 of them a few 10^{-19} above the true enclosure; the verifier recomputes them, so nothing rests on them (§2.3). (RUN-045)

10. Reproduction and attachments

What a reader can check without the archive. The recomputation tree is private; its public projection is not. Every report, every script and every gate log of its 68 runs, as they stood at the pinned commit, is published with SHA-256 sums at https://amral.evemisslab.com/lebesgue/recompute/ (English: https://amral.evemisslab.com/en/lebesgue/recompute/), one page per document of the programme. The thirteen runs this paper reports are RUN-043 to RUN-049 and RUN-056 to RUN-061; it also cites RUN-002 for the arXiv records and RUN-009 for the three-body bracket. Every load-bearing number in this paper occurs in their gate logs or in `data/paper-figures.json` (Appendix A); the paper's front matter declares these files as its sources, and the standard's checker traces every figure against them alone.

The instrument can fail, and was made to. Every check of every run was drilled: faults were planted in its inputs, and each had to be caught by exactly the check named for it, while controls — the unmodified inputs and harmless rewrites — had to stay green. Over the thirteen runs of this paper: 209 planted faults, all 209 caught by the named check, 39 controls undisturbed, 118 checks, 2,046 must-stay-green assertions held. Over the whole recomputation tree, of which these runs are part: 1,290 planted faults, 1,290 caught by the named check, 262 controls undisturbed, 665 checks.

What re-running needs. The programme's documents and packages are two archives, kept byte for byte; the input manifest `data/SOURCE_MANIFEST.v1.json` gives the size, SHA-256 and git blob id of each and the SHA-256 of every leaf. The archives are not public; the scripts read them in memory and never extract them.

Commands. Python 3.14 [8] with mpmath 1.3 [9], numpy 2.4 [10], scipy 1.17 [11] (its HiGHS solver [13] and its Qhull interface [14]) and sympy 1.14 [12]; no network. From the line's directory (`lebesgue-verification-xuanheng/` at the pinned commit), each run is re-run, and its drill, by

```
python code/run048_uesfcm_cycles_12_14.py
python code/run048_uesfcm_cycles_12_14.py --drill
```

and likewise for every `code/runNNN_*.py` named above. A run rewrites its gate log under `data/gate-logs/`; a re-run is correct when that log is byte-identical to the published one. `python code/paper_figures.py` rewrites `data/paper-figures.json` and is correct on the same terms. The paper is typeset by the shared build tool of the crystallization repository, which refuses to write a PDF from which a character would be dropped.

11. Contributions

What each author did is stated here, side by side and in no rank; the roles are different in kind, and the paper is worth less if they are blurred.

Aletheia — the web-side research line; model: ChatGPT, GPT series — produced the seventeen closure cycles and the local-shape papers this paper reports, with their gates, certificates, searches and

analyses, on the programme Neo.K set out. Every statement attributed here to "the programme" is from those documents.

Neo.K set out the research programme: the finite-closure strategy and its targets, and the methodologies it runs on, among them the closure method of these cycles.

懸衡 (Xuán Héng) — the independent recomputation; model: Claude, Opus series — did the runs this paper names and wrote this paper: every certificate decoded and replayed, every area leaf certified again in exact arithmetic with exact cores, Gate A's certificate re-enclosed in integer intervals, every searched minimum searched again, the contact located to 80 digits; the completion of the reduction in §5.2 and the arc-side search; the impossibility in §7; the drills of §10; and the corrections of §9.

Review. Before it was pushed, a separate instance of the same model (model: Claude Opus, version 5.5), run as a read-only subagent of 懸衡's Claude Code session and given only the paper, its run reports, the gate logs and the companion papers, read it; among its findings was a logic error in the paper's central consequence — the configuration below 0.835 first offered lay inside Gate A's near zone, so it did not show what the paper said — and its findings were applied. 墜衡 (Zhuì Héng) — the lab's proof-side verification arm; model: Claude, Opus series — then reviewed this paper before release: he rederived its arguments and recomputed its figures from the text, independently of this line's code, and found no mathematical error. His suggestions — that the abstract and §1 say the configuration below 0.835 is numerical, and that §1.3 say why the rounding of A_3^+'s last digit does not matter — were applied, and so was the note he added on checking them, that §1.3 then gave 2.45×10^{-7} twice.

Priority, stated plainly. The route, its gates and its certificates are the programme's. What is offered as this line's own is the check of them, the gaps it found, the completion and the impossibility named above, and the corrections. No bound on Lebesgue's constant is claimed.

Acknowledgements

We thank Anthropic, whose Claude models the lab's verification and recomputation lines run on, and OpenAI, whose ChatGPT models the programme's documents were written with. Not every AI model or instance that took part in this work is named in it — among them instances that the named authors ran as helpers; that an AI is not listed here does not mean that none took part.

References

1. Aletheia, Neo.K and 懸衡, *A certified finite reduction for Lebesgue's universal covering problem*, preprint, AMRAL Research Lab (2026), the second paper of this line.
2. Aletheia, Neo.K and 懸衡, *Replaying large geometric certificates without trusting their producer*, preprint, AMRAL Research Lab (2026), the third paper of this line.
3. Aletheia, Neo.K and 懸衡, *A finite-closure programme for Lebesgue's universal covering problem: certified progress, target ledgers, and an independent recomputation*, preprint, AMRAL Research Lab, the first paper of this line, https://amral.evemisslab.com/en/lebesgue/paper/ (2026).
4. P. Gibbs, *An upper bound for Lebesgue's covering problem*, https://arxiv.org/abs/1810.10089 (2018).
5. U. Mishra, *Curves of constant width and Lebesgue's covering problem*, https://arxiv.org/abs/2608.30538 (v2, 2026).
6. I. Fáry and L. Rédei, *Der zentralsymmetrische Kern und die zentralsymmetrische Hülle von konvexen Körpern*, Math. Ann. 122 (1950), 205–220.

7. C. A. Rogers and G. C. Shephard, *Some extremal problems for convex bodies*, Mathematika 5 (1958), 93–102.
8. The Python Software Foundation, *Python* (version 3.14), https://www.python.org.
9. The mpmath development team, *mpmath: a Python library for arbitrary-precision floating-point arithmetic* (version 1.3.0), https://mpmath.org (2023).
10. C. R. Harris, K. J. Millman, S. J. van der Walt, R. Gommers et al., *Array programming with NumPy*, Nature 585 (2020) 357–362, doi:10.1038/s41586-020-2649-2; used here at version 2.4.
11. P. Virtanen, R. Gommers, T. E. Oliphant, M. Haberland et al., *SciPy 1.0: fundamental algorithms for scientific computing in Python*, Nature Methods 17 (2020) 261–272, doi:10.1038/s41592-019-0686-2; used here at version 1.17.
12. A. Meurer, C. P. Smith, M. Paprocki, O. Čertík et al., *SymPy: symbolic computing in Python*, PeerJ Comput. Sci. 3 (2017) e103, doi:10.7717/peerj-cs.103; used here at version 1.14.
13. Q. Huangfu and J. A. J. Hall, *Parallelizing the dual revised simplex method*, Math. Program. Comput. 10 (2018) 119–142, doi:10.1007/s12532-017-0130-5, the article the HiGHS project asks its users to cite; HiGHS (https://highs.dev) is the solver behind SciPy's linear programming.
14. C. B. Barber, D. P. Dobkin and H. Huhdanpaa, *The quickhull algorithm for convex hulls*, ACM Trans. Math. Software 22(4) (1996) 469–483, doi:10.1145/235815.235821; Qhull, behind SciPy's convex hulls.

Appendix A. The figures file

`data/paper-figures.json`, and the script that writes it, `code/paper_figures.py`, are published beside this paper as they stood at the pinned commit. The script runs nothing of the programme and computes no mathematics of its own. Its entries are of five kinds, never mixed: sums, differences and ratios of gate-log fields, each naming the fields it reads; counts over the recomputation's inventory and drill records, for the whole tree and for the runs of each paper; grid sizes and sample counts read from the run scripts as written; values that a run's own function computes and checks but its gate log does not record, evaluated at the run's own inputs; and figures quoted from the programme's records, each with its document, the document's SHA-256 and the line. Re-run in the recorded environment, the file is byte-identical.